Riv. Mat. Univ. Parma (4) 3 (1977), 301-308

R. A. ALÒ, F. BERTOLINI and A. DE KORVIN (*)

A Riesz representation theorem for measures. (**)

1. - Introduction.

Let S denote a non-empty set, let Σ be a σ-algebra of subsets of S and let $ca(S, \Sigma)$ be all countably additive scalar valued measures on Σ with finite variation. If X is a Banach space, then $ca(S, \Sigma) \otimes X$ will denote *the closure in variation norm of all finite sums of the form* $\sum \mu_i x_i$, where $\mu_i \in ca(S, \Sigma)$ and $x_i \in X$. Thus $(\sum \mu_i x_i)(E) = \sum \mu_i(E) x_i$ for $E \in \Sigma$.

The main purpose of this paper is to obtain a representation theorem for linear operators T from $ca(S, \Sigma) \otimes X$ into Y, that are continuous in the variation norm. Here also Y is a Banach space. As a secondary result we characterize such operators that are compact or weakly compact.

In [7]$_1$ Mauldin characterizes the bidual of $C[0, 1]$, the space of continuous functions on the real interval $[0, 1]$. If μ is in the bidual and if T is a linear operator on this space, then this representation is given by $T(\mu) = \int \mu \, d\psi$, where the integral is defined in an appropriate manner. The techniques developed in [7]$_1$ depend strongly on T being real valued. In [7]$_2$ Mauldin represents operators on the dual space of the space $ca(S, \Sigma, X)$ of countably additive X-valued measures of finite variation. It is stated that the representation holds if and only if X is a Radon-Nikodym space.

The results in [7]$_1$ and [7]$_2$ are directly related with the results of Edwards and Wayment found in [3]. In fact the integrals of [3] and [7]$_1$ coincide on a large class of functions. In essence the results in [3] represent operators defined on point functions rather than set functions.

(*) Indirizzi degli AA.: R. A. Alò, Dept. Math., Lamar Univ., Beaumont, Texas 77710, U.S.A.; F. Bertolini, Ist. di Mat., Univ., 43100 Parma, Italy; A. De Korvin, Dept. Math., Indiana State Univ., Terre Haute, Indiana 47809, U.S.A. .

(**) The first and second Authors were partially sopported by NATO grant n. 835. — Ricevuto: 1-VI-1977.

Using techniques different than those in [**7**]$_1$ and [**7**]$_2$ but reminiscent of techniques used in [**5**] and [**6**], this article begins by showing that simple functions belong to the bidual of the space under consideration. Even when Y is the scalars, the ideas are very different from the ideas of [**7**]$_1$.

In [**3**] the notion of a *convex set function* is introduced and will be of central importance here.

We will show the following. If T is a bounded linear operator from $ca(S, \Sigma) \otimes X$ into Y, then there exists a unique convex set function ψ from Σ into $L(X, Y^{**})$, subjected to some side conditions such that $T(m) = \int m \, \mathrm{d}\psi$. Here $L(X, Y^{**})$ represents all bounded linear operators from X into the second dual Y^{**} of Y. Of course the integral will be appropriately defined. Moreover the norm of T is $\{\sup \|\psi(A)\| : A \in \Sigma\}$.

In addition we will show that, if ψ_{y^*} is an X^*-valued set function defined by $\psi_{y^*} = \langle y^*, \psi(E)\rangle$, then T is weakly compact if and only if $\{\psi_{y^*} : y^* \in \sigma^*\}$ is weakly sequentially compact. By σ^* is meant the unit ball of Y^*, the dual of Y. Also T is compact if and only if the closure of all sums of the form $\{\sum \psi(A_i)x_i : \{A_i\}$ pairwise disjoint; $\sum \|x_i\| \leqslant 1\}$ is compact in Y.

In [**1**], we have developed representation theorems for operators on measures which are absolutely continuous with respect to some fixed measure. In fact the operators in [**1**] need not be linear. There it was assumed that $T(m_1 + m_2) = T(m_1) + T(m_2)$ whenever m_1 and m_2 are concentrated on disjoint sets.

2. - Main results.

Let m be an X valued measure defined on Σ, let ψ be a set function defined on Σ with values in Y and let (,) denote a bilinear form from $X \times Y$ into Z. Throughout this paper X, Y, Z will denote Banach spaces. By $\int m \, \mathrm{d}\psi$ we mean the limit (if it exists) of $\sum \big(m(A), \psi(A)\big)$ over partitions π of S by sets from Σ. Consequently we write

$$\int m \, \mathrm{d}\psi = \lim_{\pi} \sum_{A \in \pi} \big(m(A), \psi(A)\big) .$$

In this article the bilinear form will be defined on spaces of the form $X \times L(X, Y)$, where $L(X, Y)$ denotes all linear operators from X into Y that are continuous in the norm. Thus we have $(x, u) = u(x)$.

Following [**7**]$_1$ *we assume that the cardinality of* $ca(S, \Sigma)$ *is* $2^{\aleph_0}$. Also *we assume the continuum hypothesis.*

We now quote a theorem of $[7]_1$ which will be used here. Let $\mathscr{M} = (\mu_\alpha)_{\alpha \in I}$ denote a maximal set of mutually singular measures, indexed by a set of ordinals I. Without loss of generality we may, assume $\mu_\alpha(S) = 1$ for all $\alpha \in I$.

Theorem 1 (see $[7]_1$). *The subspace $AC(S, \Sigma)$ of $ca(S, \Sigma)$ of all measures, which are absolutely continuous with respect to some finite sum from $\mathscr{M}$, is dense in the variation norm in $ca(S, \Sigma)$.*

We are now ready to give a representation theorem for linear functionals on $ca(S, \Sigma) \otimes X$. Hence we first resolve our question for the case that Y denotes the scalars.

Let $\{B_\alpha\}$ be a family of sets such that $\mu_\gamma(B_\alpha) = 0$ if $\gamma < \alpha$ and $\mu_\alpha(B_\alpha') = 0$, where B_α' denotes the complement of B_α.

Thus μ_α is concentrated on B_α. The family $\{B_\alpha\}$ is obtained as follows: let $B_{\gamma\alpha}$ be so that $\mu_\alpha(B_{\gamma\alpha}) = 0$ and $\mu_\alpha(B_{\gamma\alpha}') = 0$, let $B_\alpha = \bigcap_{\gamma<\alpha} B_{\gamma\alpha} \in \mathscr{B}_\alpha$. In $[7]_1$ it is shown that, if $B \subset B_\alpha$ and $\mu_\alpha(B) > 0$, then B does not have the same property relative to other ordinals. Let ψ be defined on Σ; ψ is μ_α-convex if

$$\psi(A \cup B) = \frac{\mu_\alpha(A)}{\mu_\alpha(A \cup B)}\, \psi(A) + \frac{\mu_\alpha(B)}{\mu_\alpha(A \cup B)}\, \psi(B)\,.$$

Theorem 2. *Let T denote a continuous linear functional on $ca(S, \Sigma) \otimes X$. Then there exists a unique set function ψ which is μ_α-convex when restricted to subsets of B_α, and such that $\psi(A) = 0$ when for no α do we have $A \subset B_\alpha$ with $\mu_\alpha(B) > 0$, and $T(r) = \int r\, d\psi$.*

Proof. Let us designate by $f\mu_\alpha(\cdot)$ the measure $\int_{(\cdot)} f d\mu_\alpha$ Since $f(\mu_{\alpha_1} + \ldots + \mu_{\alpha_n}) = f\mu_{\alpha_1} + \ldots + f\mu_{\alpha_n}$, Theorem 2 implies that finite sums of measures absolutely continuous with respect to some μ_α are dense in the variation norm in $ca(S, \Sigma)$. It follows from [4] that, if $r \ll \mu_\alpha$, then

$$r = \lim_\pi \sum_{E \in \pi} \frac{\mu_E^\alpha}{\mu_\alpha(E)}\, r(E)\,,$$

where $\mu_E^\alpha(\cdot) = \mu_\alpha(E \cap (\cdot))$. Hence

$$T(r) = \lim_\pi \sum_{E \in \pi} r(E)\, T\left[\frac{\mu_E^\alpha}{\mu_\alpha(E)}\right].$$

If $E \subset B_\alpha \in \mathscr{B}_\alpha$, set

$$\psi(E) = \frac{1}{\mu_\alpha(E)}\, T[\mu_E^\alpha]\,.$$

If E_1 and E_2 are disjoint subsets of B_α with $\mu_\alpha(E_1) > 0$, $\mu_\alpha(E_2) > 0$, then

$$\psi(E_1 \cup E_2) = \frac{T[\mu^\alpha_{E_1}]}{\mu_\alpha(E_1 \cup E_2)} + \frac{T[\mu^\alpha_{E_2}]}{\mu_\alpha(E_1 \cup E_2)} = \frac{\mu_\alpha(E_1)}{\mu_\alpha(E_1 \cup E_2)}\,\psi(E_1) + \frac{\mu_\alpha(E_2)}{\mu_\alpha(E_1 \cup E_2)}\,\psi(E_2)\,.$$

Let us now define ψ on Σ as follows:

$$\psi(B) = \begin{cases} \dfrac{T[\mu^\alpha_B]}{\mu_\alpha(B)} & \text{if } B \subset B_\alpha \text{ and } \mu_\alpha(B) > 0 \\ 0 & \text{otherwise}\,. \end{cases}$$

Clearly ψ is μ_α-convex when restricted to subsets of B_α.

Let $r = r_{\alpha_1} + \ldots + r_{\alpha_k}$, where $r_{\alpha_i} \ll \mu^i_\alpha$ and μ^i_α is one of the measures μ_α. Then

$$T(r) = \sum T(r_{\alpha_i}) \text{ and } T(r_{\alpha_i}) = \lim_\pi \sum_{E\in\pi} r_{\alpha_i}(E)\,\frac{T[\mu^{\alpha_i}_E]}{\mu_\alpha(E)} = \int r_{\alpha_i}\,d\psi\,.$$

Thus $T(r) = \int r\,d\psi$. By the density theorem stated above, we have that $T(r) = \int r\,d\psi$ for all $r \in ca(S, \Sigma) \otimes X$. This completes our proof.

Now, let $f' \in (ca(S, \Sigma) \otimes X)^*$, the dual of $ca(S, \Sigma) \otimes X$, where $ca(S, \Sigma) \otimes X$ denotes the closure in the variation norm of finite sums $\sum \mu_i x_i$ with $\mu_i \in ca(S, \Sigma)$, $x_i \in X$. If we define $\varphi^x_{f'}(\mu) = f'(\mu \cdot x)$, then it is clear that $\varphi^x_{f'}$ is a linear functional on $ca(S, \Sigma)$. Moreover, $\|\varphi^x_{f'}\| \leqslant \|f'\|\,\|x\|$. By Theorem 2 there exists a set function $\psi_{x,f'}$ defined on Σ, such that $\varphi^x_{f'}(\mu) = \int \mu\,d\psi_{x,f'}$.

We are now in a position to represent continuous linear operators on $ca(S, \Sigma) \otimes X$.

Theorem 3. *Let T be a continuous linear operator from $ca(S, \Sigma) \otimes X$ into Y. There exists a unique set function ψ from Σ into $L(X, Y^{**})$, which is μ_α-convex when restricted to subsets of B_α. In addition $\psi(A) = 0$ if for no α, $A \subset B_\alpha$ with $\mu_\alpha(B) > 0$. Moreover, $T(m) = \int m\,d\psi$ as elements of Y^{**}, and $\|T\| = \sup\{\|\psi(A)\|: A \in \Sigma\}$.*

Proof. Let $E \subset B_\alpha \in \mathscr{B}_\alpha$ with $\mu_\alpha(E) > 0$. For $x \in X$ let us define:

$$\langle \overline{\chi_E \cdot x}, f' \rangle = \mu_\alpha(E)\,\psi_{x,f'}(E)\,.$$

Since $\psi_{x,f'}$ is μ_α-convex on subsets of B_α, the above expression is well defined, that is, if $E = E_1 \cup E_2$ where E_1 and E_2 are disjoint, then $\overline{\chi_{E_1} \cdot x} + \overline{\chi_{E_2} \cdot x} =$

$= \overline{\chi_E \cdot x}$. Since $\mu_\alpha(S) = 1$ and since $\|\varphi^x_{f'}\| = \sup\{\psi_{x,f'}(A): A \in \Sigma\}$, it follows that $\overline{\chi_E x}$ is in the bidual $[ca(S, \Sigma) \otimes X]^{**}$ of $ca(S, \Sigma) \otimes X$. Now $\{E_i\}$ is a finite sequence of disjoint subsets from $\mathscr{B}_\alpha$ with $\mu_\alpha(E_i) > 0$.

$$\|\sum \overline{\chi_{E_i} x_i}\| = \sup_{\|f'\| \leqslant 1} |\sum \langle \overline{\chi_{E_i} x_i}, f' \rangle| = \sup_{\|f'\| \leqslant 1} |\sum \mu_\alpha(E_i) \psi_{x_i, f'}(E_i)|$$

$$\leqslant \sup_{\|f'\| \leqslant i} \operatorname{Max}_i \|\varphi^{x_i}_{f'}\| \mu_\alpha(S) \leqslant \operatorname{Max}_i \|x_i\| .$$

Let us proceed now to define ψ from Σ into $L(X, Y^{**})$ as follows:

$$\psi(E)x = \begin{cases} \dfrac{T'' \, \overline{[\chi_E x]}}{\mu_\alpha(E)} & \text{if } E \subset B_\alpha \in \mathscr{B}_\alpha \text{ and } \mu_\alpha(E) > 0 \\ 0 & \text{otherwise}. \end{cases}$$

We are denoting by T' and T'' the adjoint and double adjoint, respectively, of T.

Recall that $\sum \mu_i x_i$ are dense in $ca(S, \Sigma) \otimes X$. By the theorem of $[7]_1$, we may assume $\mu_i \ll \mu^i_\alpha$, where μ^i_α is one of the μ_α. We note that it is possible for $\mu^i_\alpha = \mu^j_\alpha$ when $i \neq j$.

Now $\langle T(\sum \mu_i x_i), y' \rangle = \langle \sum \mu_i x_i, t' y' \rangle = \sum \langle \mu_i x_i, T' y' \rangle = \sum \int \mu_i \, d\psi_{x_i, T'y'}$. If $E \subset B_\alpha$, $\mu_\alpha(E) > 0$, then $\langle T''(\overline{\chi_E x}), y' \rangle = \langle \chi_E x, T' y' \rangle = \mu_\alpha(E) \psi_{x, T'y'}(E)$. Also $\langle T''(\overline{\chi_E x}), y' \rangle = \langle \mu_\alpha(E) \psi(E) x, y' \rangle$. Hence: $\langle \psi(E) x, y' \rangle = \psi_{x, T'y'}(E)$.

Thus: $\langle T(\sum \mu_i x_i), y' \rangle = \sum \int \mu_i \, d\langle \psi(\cdot) x_i, y' \rangle$ when $\mu_i \ll \mu^i_\alpha$. Now

$$\int \mu_i \, d\langle \psi(\cdot) x_i, y' \rangle = \lim_\pi \sum_{E \in \pi} \langle \mu_i(E) \psi(E) x_i, y' \rangle$$

$$= \lim_\pi \sum_{E \in \pi} \langle \psi(E) \mu_i(E) x_i, y' \rangle = \lim_\pi \sum_{E \in \pi} \psi(E) \mu_i(E) x_i, y' \rangle .$$

By [4], μ_i is the limit in variation norm of $\sum_{E \in \pi} \big(\mu^i_\alpha[En(\cdot)]/\mu^i_\alpha(E)\big) \; \mu_i(E)$ as π is refined. Let K be any μ^i_α-convex set function from $\sum$ into $L(X, Y^{**})$ which in uniformly bounded in norm. Since it is easy to check that $\sum_{E \in \pi} \big(\mu^i_\alpha[En(\cdot)]/\mu^i_\alpha(E)\big)$ are integrable with respect to K, it follows that μ_i is. Thus

$$\lim_\pi \sum_{E \in \pi} \mu_i(E) \psi(E) x_i$$

exists in the norm of Y^{**}. Thus

$$\int \mu_i \, d\langle \psi(\cdot) x_i, y' \rangle = \langle \int \mu_i x_i \, d\psi, y' \rangle , \qquad \langle T(\sum \mu_i x_i), y' \rangle = \langle \int \sum \mu_i x_i \, d\psi, y' \rangle .$$

Thus

$$T(m) = \int m \, d\psi \qquad \text{for } m \in ca(S, \sum) \otimes X .$$

It is clear from the definition that ψ is μ_α-convex on subsets of B_α and $\psi(A) = 0$ if for no α, $A \subset B_\alpha$ with $\mu_\alpha(A) > 0$.

We now show that ψ is unique. Let ψ' be a set function satisfying the conditions of Theorem 3. Assume $T(m) = \int m \, d\psi'$. Let us set $m = W_A \cdot x$, where $w_A(\cdot) = (\mu_\alpha(A \cap (\cdot)))/\mu_\alpha(A)$ for $A \subset B_\alpha \in \mathscr{B}_\alpha$. Thus we see that $\|\psi'(A)\| \leqslant \|T\|$.

Now $\langle T(\mu \cdot x), y' \rangle = \int \mu \, d\langle \psi'(\cdot)x, y' \rangle$. In fact $\langle \int (\mu \cdot x) \, d\psi', y' \rangle = \langle \lim_\pi \sum_{E \in \pi} \mu(E) \cdot \psi'(E)x, y' \rangle$. As above it may be shown that μ is integrable with respect to $\psi'(\cdot)x$. Thus

$$\langle \int \mu \cdot x \, d\psi', y' \rangle = \lim_\pi \langle \sum_{E \in \pi} \mu(E) \langle \psi'(E)x, y' \rangle \rangle = \int \mu \, d\langle \psi'(x \cdot)x, y' \rangle .$$

Now

$$\langle T(\mu \cdot x), y' \rangle = \langle \mu \cdot x, T'y' \rangle = \int \mu \, d\psi_{x, T'y'} = \int \mu \, d\langle \psi(\cdot)x, y' \rangle .$$

So

$$\int \mu \, d\langle \psi'(\cdot)x, y' \rangle = \int \mu \, d\langle \psi(\cdot)x, y' \rangle .$$

By setting $\mu = W_A$ with $A \subset B_\alpha \in \mathscr{B}_\alpha$ and by using the μ_α-convexity of ψ and ψ', it can be shown that $\int W_A \, d\langle \psi'(\cdot)x, y' \rangle = \langle \psi'(A)x, y' \rangle$ and $\int W_A \, d\langle \psi(\cdot)x, y' \rangle = \langle \psi(A)x, y' \rangle$. Hence it follows that $\psi = \psi'$.

By definition of $\int \mu \, dK$ it is obvious that $\|T\| \leqslant \sup_{A \in \Sigma} \|\psi(A)\|$. Thus $\|T\| = \sup_{A \in \Sigma} \|\psi(A)\|$. This completes our proof.

Let m be a measure from Σ into X. We say that m has an *approximate Radon-Nikodym derivative* if for every $\varepsilon > 0$ there exists a set function σ from Σ into X of the form $\sigma = \sum_{k=1}^{n} |m|_{A_k} x_k$, where A_k are disjoint sets of Σ and $|m|_{A_k}$ is the contraction of the variation of m to A_k and $\operatorname{var}[m - \sigma] < \varepsilon$.

Let $ca(S, \Sigma, X)$ *denote X-valued countably additive measures of finite variation.*

Let us recall also that X has *the Radon-Nikodym property* if every X-valued countably additive set function of finite variation, which is absolutely continuous relative to a positive measure of finite variation, has a density (X-valued) with respect to that measure. For example, reflexive and separable dual spaces have that property.

Corollary. *If X is a Radon-Nikodym space and T is a continuous linear operator from $ca(S, \Sigma, X)$, then T admits the representation of Theorem 3. Moreover X has the Radon-Nikodym property if and only if $ca(S, \Sigma, X) = ca(S, \Sigma) \otimes X$. If the above representation holds for all spaces Y, then X has the Radon-Nikodym property.*

Proof. Now if X is a Radon-Nikodym space then every X-valued additive set function has the approximate Radon-Nikodym derivative. Thus $\{\sum \mu_i x_i\}$ is dense in $ca(S, \Sigma, X) = ca(S, \Sigma) \otimes X$. Conversely if $ca(S, \Sigma, X) = ca(S, \Sigma) \otimes X$, then the above representation holds for operators from $ca(S, \Sigma, X)$ into X^*. It is pointed out in [7]$_2$ that this implies that X has the Radon-Nikodym property. This completes the proof of our Corollary.

We now proceed to characterize compact and weakly compact operators. To this end let $co(S, \Sigma, Z)$ *denote all set functions from Σ into Z which are μ_α-*convex *when restricted to B_α*, and which are zero on sets A such that for no α, $A \subset B_\alpha$ with $\mu_\alpha(A) > 0$. If ψ is as in Theorem 3, let $\langle \psi_{y^*}(A), x\rangle = \langle y^*, \psi(A)x\rangle$. Thus $\psi_{y^*} \in co(S, \Sigma, X^*)$ and $co(S, \Sigma, Z)$ is a normed space with $\|\psi\| = \sup\{\|\psi(A)\| : A \in \Sigma\}$. Finally let σ^* denote *the unit ball* of Y^*.

Theorem 4. *The operator T is weakly compact if and only if $\{\psi_{y^*} : y^* \in \sigma^*\}$ is weakly sequentially compact in $co(S, \Sigma, X^*)$. It is compact if and only if $\{\sum \psi(A_i)x_i : \sum \|x_i\| \leqslant 1, A_i \in \Sigma, \{A_i\}$ pairwise disjoint$\}$ is precompact in Y.*

Proof. Now T is weakly compact if and only if T^* is (see [2]). In addition

$$\langle T^*(y^*), m\rangle = \langle y^*, Tm\rangle = \int m \, d\psi_{y^*}.$$

Thus $T^*(y^*) = \psi_{y^*}$. Since $\|\psi_{y^*}(A)\| \leqslant \|\psi(A)\|$ whenever $y^* \in \sigma^*$, one has $\psi_{y^*} \in co(S, \Sigma, X^*)$. By the Eberlein-Smulian Theorem (see [2]), T is weakly compact if and only if $T^*\sigma^*$ is weakly sequentially compact.

Since $W_{A\cup B} = (\mu_\alpha(A)/\mu_\alpha(A \cup B))\, W_A + (\mu_\alpha(B)/\mu_\alpha(A \cup B))\, W_B$ for A and B disjoint subsets of $B_\alpha \in \mathscr{B}_\alpha$, it follows that the unit ball in $ca(S, \Sigma) \otimes X$ is the closure in variation norm of measures of the form $\sum W_{A_i} x_i$ where $A_i \subset B^i_\alpha$, $\{A_i\}$ are pairwise disjoint and $\sum \|x_i\| \leqslant 1$. Now T is compact if and only if the image of the unit ball of $ca(S, \Sigma) \otimes X$ is precompact in Y, that is if and only if $\{\sum \psi(A_i)x_i : \sum \|x_i\| \leqslant 1, \{A_i\}$ pairwise disjoint in $\Sigma\}$ is precompact in Y.

Bibliography.

[1] R. A. Alò and A. de Korvin, *Integration of measures*, (to appear).

[2] N. Dunford and J. T. Schwartz, *Linear operators*, Vol. 1, New York, 1958.

[3] J. R. Edwards and S. Wayment, *Extensions of the v-integral*, Trans. Amer. Math. Soc. **191** (1974), 1-20.

[4] C. Fefferman, *A Radon-Nikodym theorem for finitely additive set functions*, Pacific J. Math. **23** (1967), 35-45.

[5] R. K. Goodrich, *A Riesz representation theorem*, Proc. Amer. Math. Soc. **24** (1970), 629-636.

[6] A. de Korvin and R. J. Easton, *Some representation theorems*, Rocky Mountain J. Math. (3) **1** (1971), 561-573.

[7] R. D. Mauldin: $[7]_1$ *A representation theorem for the second dual of* $C[0, 1]$, Studia Math. **46** (1973), 197-200; $[7]_2$ *The continuum hypothesis, integration and duals of measure spaces*, (to appear).

Abstract.

Let S denote a non-empty set, let Σ be a σ-algebra of subsets of S and let $ca(S, \Sigma)$ be all countably additive scalar valued measures on Σ with finite variation. If X is a Banach space, then $ca(S, \Sigma) \otimes X$ will denote the closure in variation norm of all finite sums of the form $\sum \mu_i x_i$, *where* $\mu_i \in ca(S, X)$ *and* $x_i \in X$. *Thus* $(\sum \mu_i x_i)(E) = \sum \mu_i(E) x_i$ *for* $E \in \Sigma$. *The main purpose of this paper is to obtain a representation theorem for linear operators T from $ca(S, \Sigma) \otimes X$ into Y that are continuous in the variation norm. Here also Y is a Banach space. As a secondary result, we characterize such operators that are compact or weakly compact.*

* * *