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ALESSANDRO BELCASTRO and MARCO MARGIOCCO (*)

$S\theta$-symmetrizations and $C\theta$-categories. (**)

1. - Introduction.

This paper treats of some particular symmetrizations of categories. We recall that a symmetrization of a category $\mathscr{C}$ is an embedding of $\mathscr{C}$ in an involution category $\mathscr{H}$ having the same objects. A precise definition of symmetrization can be found in $[\mathbf{4}]_1$.

It is well known that each abelian category $\mathscr{A}$ can be embedded in an involution category $\tilde{\mathscr{A}}$ (the category of relations, or correspondences of $\mathscr{A}$) having the same objects, where the morphisms from A to B are the subobjects of $A \times B$ (Mac Lane $[\mathbf{6}]_1$, $[\mathbf{6}]_2$; Hilton [**5**]; Brinkmann $[\mathbf{1}]_2$). This embedding in an involution category can be generalized to exact categories (conjecture of Puppe [**7**], proved by Calenko $[\mathbf{3}]_1$, $[\mathbf{3}]_2$, Brinkmann $[\mathbf{1}]_1$, Brinkmann and Puppe [**2**]).

In this way we obtain the canonical symmetrization $s: \mathscr{E} \to \mathscr{H}$ of an exact category $\mathscr{E}$.

It has been proved ($[\mathbf{4}]_7$, 1.10) that the category $\mathscr{H}$ is orthodox iff $\mathscr{E}$ has distributive lattices of subobjects. We recall that a regular involution category is called orthodox iff the composition of idempotent endomorphisms is idempotent $[\mathbf{4}]_3$.

Orthodoxy of $\mathscr{H}$ is a necessary and sufficient condition in order that canonical isomorphisms between subquotients of $\mathscr{E}$ should be composable ($[\mathbf{4}]_5$, 3; 17). This fact allows us to define induced relations between subquotients which are compatible with composition ($[\mathbf{4}]_5$, 3.9).

We can also quotient $\mathscr{H}$ by a congruence of category Φ such that canoni-

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(*) Indirizzo: Ist. Mat. Università, Via L. B. Alberti 4, I-16132 Genova, Italy.
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cally isomorphic subquotients of $\mathscr{E}$ become the same subobject in $\mathscr{H}/\Phi$. Moreover, the composition

$$\mathscr{E} \xrightarrow{i} \mathscr{H} \xrightarrow{p} \mathscr{H}/\Phi$$

is an inverse symmetrization of $\mathscr{E}$, that is $\mathscr{H}/\Phi$ is an inverse category $[4]_3$.

More generally, this question has been treated for a « quaternary category » $\mathscr{E}$ $[4]_2$. In this case we can construct the quaternary symmetrization $s\colon \mathscr{E} \to \mathscr{H}$, generalising the category of relations of an exact category, and also Brinkmann's proceeding $[1]_1$. Then $\mathscr{H}$ is orthodox iff $\mathscr{E}$ is orthoquaternary $[4]_7$, i.e. $\mathscr{E}$ is quaternary and satisfies the two axioms $C\theta\,4$, $C\theta\,4^*$ (**2.1**).

In this paper we construct directly particular inverse symmetrizations, which we call $S\theta$-symmetrizations, without passing through quaternary symmetrizations. $S\theta$-symmetrizations are defined for a class of categories larger than that of orthoquaternary categories, which we call $C\theta$-categories. In the orthoquaternary case the $S\theta$-symmetrization coincides with the symmetrization $s = pi\colon \mathscr{E} \to \mathscr{H}/\Phi$ above mentioned.

In **1** we define $S\theta$-symmetrizations by axioms $S\theta\,1$-7.

In **2** we give axioms for $C\theta$-categories (among them the above $C\theta\,4$, $C\theta\,4^*$).

In **3** we prove: (a) a category $\mathscr{C}$ has a $S\theta$-symmetrization iff $\mathscr{C}$ is a $C\theta$-category; (b) the $S\theta$-symmetrization of a $C\theta$-category is unique; moreover it can be obtained using the symmetrizer θ (**3.5**). This symmetrizer can be applied to any category, but it gives $S\theta$-symmetrizations only if it is applied to $C\theta$-categories. Finally we construct explicitly the symmetrized category.

In **4** we give an example of a $C\theta$-category which is not quaternary, so that its $S\theta$-symmetrization cannot be obtained from a quaternary symmetrization.

All the proofs are given in **5**.

1. - $S\theta$-symmetrizations.

1.1. – Let $s\colon \mathscr{C} \to \mathscr{H}$ be a symmetrization of a category $\mathscr{C}$ $[4]_1$; let us consider the following conditions $S\theta\,1$-7.

$S\theta\,1$: $\mathscr{C}$ is a factorizing category, i.e. any morphism a has an epic-monic factorization $a = m \circ p$ (where p is epic, m is monic) unique up to isomorphism, called *the canonical factorization of a.*

$S\theta\,2$: for any pair (m, p) of morphisms of $\mathscr{C}$, m monic, p epic, having the same codomain, there are morphisms m', p' (m' monic, p' epic) such that $mp' = pm'$ (we call this property *existence of a lower commuting*).

$S\theta$ 3: the functor s is faithful and preserves monics and epics. (s being faithful, we can identify any morphism u of $\mathscr{C}$ with its image $s(u)$ and $\mathscr{C}$ with the subcategory $s(\mathscr{C})$ of $\mathscr{H}$).

$S\theta$ 4: $\mathscr{H}$ is a regular involution category, i.e. for each morphism α, we have $\alpha\tilde{\alpha}\alpha = \alpha$ (equivalently $\tilde{\alpha}\alpha\tilde{\alpha} = \tilde{\alpha}$).

$S\theta$ 5: $\mathscr{H}$ is an inverse category, i.e. for each morphism α of $\mathscr{H}$ there is a unique morphism β such that $\alpha\beta\alpha = \alpha$ and $\beta\alpha\beta = \beta$; note that by $S\theta$ 4 we have necessarily $\beta = \tilde{\alpha}$.

$S\theta$ 6: s has quaternary factorizations, i.e. each morphism α of $\mathscr{H}$ has a factorization (not necessarily unique) $\alpha = n\tilde{q}p\tilde{m}$, where m and n are monics of $\mathscr{C}$, p and q epics of $\mathscr{C}$.

Note that by $S\theta$ 1-3-4-6 we have: (a) a quaternary factorization of a morphism α of $\mathscr{H}$ yields an epic-monic factorization in $\mathscr{H}$ $\alpha = (n\tilde{q})(p\tilde{m})$ which is necessarily unique by $S\theta$ 4. Therefore $\mathscr{H}$ is a factorizing category; (b) if $\alpha = n\tilde{q}p\tilde{m}$ is a quaternary factorization of an isomorphism α of $\mathscr{H}$, then m, p q, n are isomorphisms of $\mathscr{C}$ ($[1]_2$, 16.1); consequently s is invariant (i.e. $\mathscr{C}$ and $\mathscr{H}$ have the same isomorphisms); (c) if $\alpha = n\tilde{q}p\tilde{m}$ is a quaternary factorization of α, α is epic in $\mathscr{H}$ iff n and q are isomorphisms in $\mathscr{C}$, α is monic iff m and p are isomorphisms in $\mathscr{C}$. s being invariant, the canonical factorization $\alpha = (n\tilde{q})\cdot(p\tilde{m}) = \mu\pi$ is unique up to isomorphism of $\mathscr{C}$. This allows us to define the sets $\mathscr{H}_d$, $\mathscr{H}_{d*}$, $\mathscr{H}_v$, $\mathscr{H}_{v*}$ of morphisms of $\mathscr{H}$ in the following way

$$\alpha \in \mathscr{H}_d \Leftrightarrow \pi \in \mathscr{C}, \qquad \alpha \in \mathscr{H}_{d*} \Rightarrow \mu \in \mathscr{C},$$
$$\alpha \in \mathscr{H}_v \Leftrightarrow \tilde{\mu} \in \mathscr{C} \qquad (\Leftrightarrow \tilde{\alpha} \in \mathscr{H}_d),$$
$$\alpha \in \mathscr{H}_{v*} \Leftrightarrow \tilde{\pi} \in \mathscr{C} \qquad (\Leftrightarrow \tilde{\alpha} \in \mathscr{H}_{d*}).$$

It is obvious that if $\alpha = n\tilde{q}p\tilde{m} = \mu\pi$ ($\pi = p\tilde{m}$, $\mu = n\tilde{q}$):

$$\alpha \in \mathscr{H}_d \Leftrightarrow \alpha = n\tilde{q}\pi\tilde{1} \quad \text{is a quaternary factorization},$$
$$\alpha \in \mathscr{H}_{d*} \Leftrightarrow \alpha = \mu\tilde{1}p\tilde{m} \quad \text{is a quaternary factorization},$$
$$\alpha \in \mathscr{H}_v \Leftrightarrow \alpha = 1\tilde{\tilde{\mu}}p\tilde{m} \quad \text{is a quaternary factorization},$$
$$\alpha \in \mathscr{H}_{v*} \Leftrightarrow \alpha = n\tilde{q}1\tilde{\tilde{\pi}} \quad \text{is a quaternary factorization}.$$

$S\theta$ 7: $\mathscr{H}_d$, $\mathscr{H}_{d*}$, $\mathscr{H}_v$, $\mathscr{H}_{v*}$ are subcategories of $\mathscr{H}$ (it is sufficient to verify this for $\mathscr{H}_d$, $\mathscr{H}_{d*}$).

We call *$S\theta$-symmetrization* a symmetrization s verifying axioms $S\theta$ 1-7.

1.2. – We recall the following lemma ($[4]_6$, 5.3): « In an inverse category $\mathscr{H}$ a square of monics is bicommutative iff it is anticommutative, iff it is a pullback; an epic-monic square is commutative iff it is bicommutative, i.e. if π, π' are epics μ, μ' are monics and $\mu\pi' = \pi\mu'$, then $\mu'\tilde{\pi}' = \tilde{\pi}\mu$ ».

1.3. – Lemma. *Let $\mathscr{C}$ be a category having a $S\theta$-symmetrization $s: \mathscr{C} \to \mathscr{H}$. Then*: (a) *$\mathscr{C}$ and $\mathscr{H}$ have* (*finite*) *intersections of subobjects*; (b) *the* (*finite*) *intersections of subobjects of $\mathscr{C}$ are the same in $\mathscr{C}$ and in $\mathscr{H}$; i.e. a square of monics of $\mathscr{C}$ is a pullback in $\mathscr{C}$ iff it is a pullback in $\mathscr{H}$* (1) (see Proof. **5.1**).

1.4. – Dually, an analogous lemma is true for intersections of quotients and pushouts of epics.

2. - $C\theta$-categories.

2.1. – Let us consider the following conditions for a category $\mathscr{C}$.

$C\theta\,1$: $\mathscr{C}$ is factorizing (identical to $S\theta\,1$).

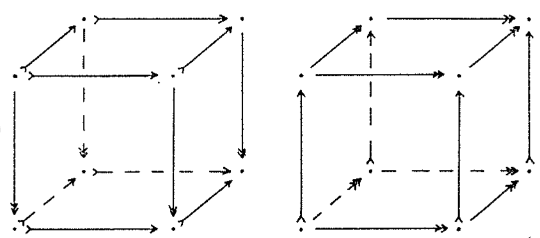
$C\theta\,2$: identical to $S\theta\,2$ (existence of *a lower commuting* (**1.1**, $S\theta\,2$)).

$C\theta\,3$: $\mathscr{C}$ has pullbacks of monics (finite intersections of subobjects).

$C\theta\,3^*$: $\mathscr{C}$ has pushouts of epics (finite intersections of quotients).

$C\theta\,4$: if the diagram (1) is commutative and its upper square is a pullback, so is the lower one. If $C\theta\,1$ is verified, $C\theta\,4$ is equivalent to the statement that the direct image of subobjects preserves (finite) intersections.

$C\theta\,4^*$: if the diagram (1*) is commutative and its upper square is a pushout, so id the lower one. If $C\theta\,1$ is verified, $C\theta\,4^*$ is equivalent to the statement that the inverse image of quotients preserves (finite) intersections.

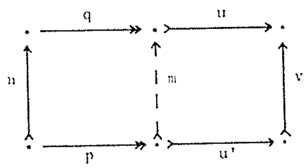
(1)  (1*)

(1) As $\mathscr{C}$ and $\mathscr{H}$ are factorizing, in both categories the lattice-definition of intersection of subobjects (given by a pullback in the subcategory of monics) is equivalent to the one given by a pullback in $\mathscr{C}$ (or $\mathscr{H}$).

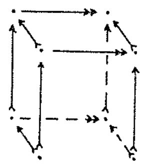
2.2. – We call *$C\theta$-category* a category verifying the axioms $C\theta\,1$ ... $C\theta\,4^*$.

2.3. – Lemma. *In a category verifying $C\theta\,1$, if the outer rectangle of the diagram* (1) *is commutative, there is a unique morphism m such that inner squares are commutative. Moreover, m is monic* (see Proof. **5.2**).

(1)

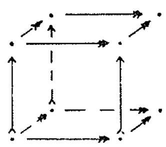


2.4. – Lemma. *Let $\mathscr{C}$ be a $C\theta$-category. If in the commutative diagram*



the upper and lower squares are pullbacks, there is a (unique) epic completing the projection of the upper square on the lower one (see Proof **5.3**).

2.5. – Lemma (dual of **2.4**). *Let $\mathscr{C}$ be a $C\theta$-category. If in the commutative diagram*



the upper and lower squares are pushouts, there is a (unique) monic completing the injection of the lower square in the upper one (proof dual of **5.3**).

It can be easily proved that, if $C\theta\,1$ and $C\theta\,3$ are verified, Lemma **2.4** is equivalent to axiom $C\theta\,4$, and dually for Lemma **2.5**.

2.6. – Lemma. *Let $\mathscr{C}$ be a $C\theta$-category, m a monic of $\mathscr{C}(A, B)$, p and q morphisms of $\mathscr{C}(B, C)$. If pm and qm are equal epics, then p and q are equal epics* (see Proof **5.4**).

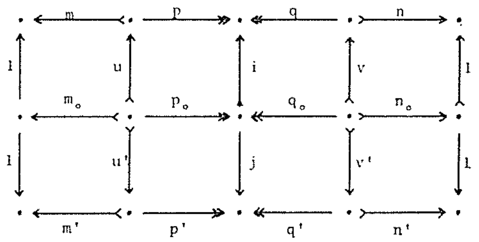
3. - Existence and uniqueness of the $S\theta$-symmetrization of a $C\theta$-category.

3.1. – Theorem. *A category $\mathscr{C}$ has a $S\theta$-symmetrization iff it is a $C\theta$-category.*

For the necessity see **5.5**. For the sufficiency, we must construct directly the involution category $\mathscr{H}$ and the $S\theta$-symmetrization $s\colon \mathscr{C} \to \mathscr{H}$. This construction is described in **3.2**, **3.3**, **3.4**.

3.2. – Lemma. *Let $s\colon \mathscr{C} \to \mathscr{H}$ be an $S\theta$-symmetrization and let $\alpha = n\tilde{q}p\tilde{m}$, $\beta = n'\tilde{q}'p'\tilde{m}'$ be quaternary factorizations of two morphisms α and β of $\mathscr{H}$. Then $\alpha = \beta$ iff there exists an «intermediate» morphism $\gamma = n_0\tilde{q}_0 p_0 \tilde{m}_0$ and «vertical» morphisms u, v, u', v', i, j such that the following diagram of $\mathscr{C}$ is commutative*

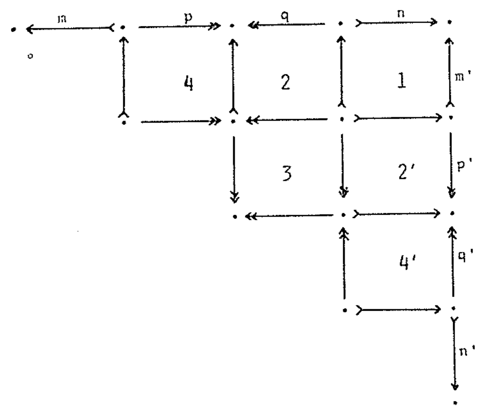
(1)



where u, u', v, v' are (necessarily) monics, i and j are isomorphisms. Moreover if $\alpha = \beta$ we can choose the morphisms u, u', v, v' so that the squares (m, u, u', m') and (n, v, v', n') are pullbacks (see Proof **5.6**).

3.3. – Lemma. *Let $s\colon \mathscr{C} \to \mathscr{H}$ be a $S\theta$-symmetrization and let $\alpha = n\tilde{q}p\tilde{m}$, $\beta = n'\tilde{q}'p'\tilde{m}'$ be quaternary factorizations of two composable morphisms α and β of $\mathscr{H}$. A quaternary factorization of $\beta\alpha$ is given by the lower path of the following diagram (making the 4 possible compositions)*

(1)



where the square (1) *is a pullback* ($C\theta$ 3); (2) *and* (2′) *are commutative* ($C\theta$ 1); (3) *is a pushout* ($C\theta$ 3*); (4) *and* (4′) *are commutative* ($C\theta$ 2).

The proof follows at once, the squares (2) and (2′) being commutative and the squares (1), (3), (4), (4′) being bicommutative by **1.2**.

3.4. – Let $\mathscr{C}$ be a $C\theta$-category. We define the involution category $\mathscr{H}$ specifying its objects, morphisms and compositions using **3.2** and **3.3**. Let the objects of $\mathscr{H}$ be the same of the objects of $\mathscr{C}$; let the elements of $\mathscr{H}(A, B)$ be the equivalence classes of chains of morphisms of $\mathscr{C}$ of the type

$$A \xleftarrow{\;m\;} \cdot \xrightarrow{\;p\;} \cdot \xleftarrow{\;q\;} \cdot \xrightarrow{\;n\;} B$$

modulo the following equivalence relation: two chains $C = (m, p, q, n)$ and $C' = (m', p', q', n')$ are equivalent if there exist an *intermediate* chain $C'' = (m_c, p_0, q_0, n_0)$ and *vertical* morphisms u, v, u', v', i, j such that the diagram **3.2** (1) is commutative, where i and j are isomorphisms and u, v, u', v' morphisms which are necessarily monics. This is an equivalence relation (see Proof **5.7**).

To define the product of two composable morphisms α and β of $\mathscr{H}$ represented by the chains $C = (m, p, q, n)$ and $C' = (m', p', q', n')$ we use the diagram **3.3** (1). In this diagram the commutative squares (4) and (4′) are obtained by $C\theta$ 2, hence they are not necessarily unique, not even up to isomorphism.

We define $\beta\alpha$ as the morphism of $\mathscr{H}$ represented by the *lower path* of the diagram, when the 4 possible compositions are made. This definition does not depend neither on the choice of the chains representing α and β nor on the choices made to construct the squares (4) and (4′) (see Proof **5.8**). The composition thus defined is associative (see Proof **5.9**).

If 1_A is the morphism of $\mathscr{H}$ represented by the chain $(1_A, 1_A, 1_A, 1_A)$ of identities of $\mathscr{C}$, we have obviously: $\alpha 1_A = \alpha$ and $1_A\beta = \beta$ whenever the compositions are defined. If α is an element of $\mathscr{H}(A, B)$ represented by the chain (m, p, q, n), we define $\tilde{\alpha}$ as the element of $\mathscr{H}(B, A)$ represented by the chain (n, q, p, m). Clearly, the definition of $\tilde{\alpha}$ does not depend on the choice of the chain representing α; with this definition $\mathscr{H}$ is an involution category ($\tilde{\alpha}\tilde{\beta} = (\beta\alpha)^{\sim}$ being obvious).

The function $s\colon \mathscr{C} \to \mathscr{H}$, mapping any object in itself and any morphism $a = np$ (canonical factorization in $\mathscr{C}$) in the equivalence class of the chain $(1, p, 1, n)$ (evidently well defined), is a symmetrization of $\mathscr{C}$ (see Proof **5.10**) and satisfies the axioms $S\theta$ 1-7 (see Proof **5.11**). So each $C\theta$-category has a $S\theta$-symmetrization.

3.5. – Theorem. (Uniqueness of the $S\theta$-symmetrization). *All the $S\theta$-symmetrizations of a $C\theta$-category $\mathscr{C}$ are isomorphic to the symmetrization $s_{\mathscr{C}\theta}$: $\mathscr{C} \to \mathscr{C}^0$, obtained by the symmetrizer θ, associated to the following square types* ($[4]_1$, **2.8** and **4**): *pullbacks of monics*; *pushouts of epics*; *mixed commutative squares, i.e squares of the tipe (m, p, m', p'), where m, m' are monics, p, p' are epics and $pm = m'p'$* (see Proof **5.12**).

4. - Examples.

4.1. – Any inverse involution category $\mathscr{H}$ is a $C\theta$-category; its $S\theta$-symmetrization is the identical functor, so its symmetrized category is $\mathscr{H}$ itself (as we can easily see).

4.2. – Example of a $C\theta$-category which is not quaternary. The category $\mathscr{G}^{\#0}$ (θ-symmetrized of $\mathscr{G}^{\#}$, distributive expansion of the category of abelian groups $[4]_8$) is a $C\theta$-category (being an inverse involution category) but is not quaternary: in order to prove it, we demonstrate that there is a pair of morphisms (m, p), m monic, p epic with the same codomain, having no *epic-monic* pullback, i.e. no pullback of the type (m, p, m', p'), where m' is monic, p' epic and $mp' = pm'$.

Let A be an abelian group having a proper filtration $0 \subsetneq K \subsetneq H \subsetneq A$. Let us consider (as epic-monic pair) the pair of morphisms μ and $\tilde{\nu}$ (diagram 1), where μ and ν are the equivalence classes in $\mathscr{G}^{\#0}$ of the canonical inclusions $\mu: 0 \to H/K$ and $\nu: H/K \to A$. A is provided with the distributive lattice of subobjects supplied by its filtration, while H/K with the lattice consisting of the null and total subobjects.

Let us suppose that it exists the pullback $\tilde{\mu}'$, ν' of μ, $\tilde{\nu}$ (where μ', ν' are monics) and let X be the domain of ν'. Then X must be isomorphic to one of the following objects: 0, H, K, A, A/K, A/H, H/K. By **1.2** the square μ, ν, μ', ν' is a pullback. Hence X can not be H, A, H/K, A/K, otherwise instead of the object O we should have the object H/K.

For each object Y and each pair of morphisms $\alpha: Y \to O$ and $\beta: Y \to A$ such that $\mu\alpha = \tilde{\nu}\beta$ there exists a morphism γ such that $\alpha = \tilde{\mu}'\gamma$ and $\beta = \nu'\gamma$. If we choose Y depending on X according to the scheme

X	Y
O	K
K	A/H
A/H	K

and if α is the canonical projection and β the canonical inclusion, then γ is a monic. Then, if $X = S/T$ and $Y = S'/T'$, we must have: $S \subset S' + T$, $S \cap T' \subset T$. We can easily verify that this leads in each case to a contradiction.

An analogous example can be made for any exact category $\mathscr{E}$ having an object A with a proper filtration $0 \subsetneq K \subsetneq H \subsetneq A$. Then the category $\mathscr{E}^{\#0}$ has not *epic-monic* pullbacks, so it gives an example of a $C\theta$-category which is not quaternary.

5. - Proofs.

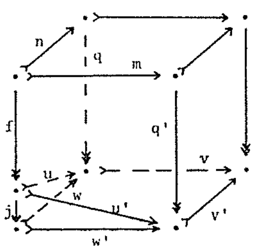
5.1. - Proof of **1.3**. Since $\mathscr{H}$ is an inverse category ($S\theta\,5$), the subcategory of monics of $\mathscr{H}$ has pullbacks $([4]_6, 5.8)$. For the $\mathscr{C}$ case, let m, n be *converging* monics of $\mathscr{C}$; by $S\theta\,3$ m, n are also monics of $\mathscr{H}$, hence they have a pullback (μ, ν, m, n) in $\mathscr{H}$, which is bicommutative by **1.2**, i.e. $m\nu = n\mu$, $\tilde{n}m = \mu\tilde{\nu}$. By the invariance of s, we have: $m \in \mathscr{H}_{dd^*\nu^*}$ (here $\mathscr{H}_{dd^*\nu^*}$ means $\mathscr{H}_d \cap \mathscr{H}_{d^*} \cap \mathscr{H}_{\nu^*}$) $\tilde{m} \in \mathscr{H}_{d^*\nu\nu^*}$ then, by $S\theta\,7$, $\mu\tilde{\nu} = \tilde{n}m \in \mathscr{H}_{d^*\nu^*}$. This means that $\mu, \nu \in \mathscr{C}$.

Now we prove that (μ, ν, m, n) is a pullback in the subcategory of monics of $\mathscr{C}$.

Let m', n' be monics of $\mathscr{C}$ such that $mn' = nm'$. Then it exists a unique monic μ' of $\mathscr{H}$ such that $\nu\mu' = n'$, $\mu\mu' = m'$. We have necessarily: $\mu' = \tilde{\nu}n' = \tilde{\mu}m'$. From $n' \in \mathscr{H}_{dd^*\nu^*}$, $\tilde{\nu} \in \mathscr{H}_{d^*\nu\nu^*}$ it follows that $\mu' = \tilde{\nu}n' \in \mathscr{H}_{d^*\nu^*} \Rightarrow \mu' \in \mathscr{C}$. So each pair m, n of converging monics of $\mathscr{C}$ has a pullback in $\mathscr{C}$, which is also a pullback in $\mathscr{H}$.

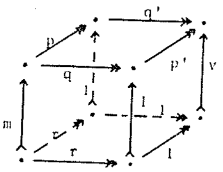
5.2. - Proof of **2.3**. In the diagram **2.3** (1) let $m_0 p_0$ be a canonical factorization of qn. Then $(um_0)p_0 = (vu')p$ are canonical factorizations of the *diagonal* morphism, so there exists a unique isomorphism i such that $p_0 = ip$. The monic $m = m_0 i$ (unique since p is epic) is the required one.

5.3. - Proof of **2.4**.



Let $u'p$ be the canonical factorization of $q'm$; by **2.3** it exists a unique monic u such that the diagram is commutative. By $C\theta$ 4 the face (u, v, u', v') is a pullback; since also the face (w, v, w', v') is a pullback, it exists the isomorphism j such that the diagram is commutative. The required epic is jp (unique because vw is monic).

5.4. – Proof of **2.6**. Since $pm = qm$ is epic, p and q are epics. Let $r = pm = qm$, and consider the commutative diagram



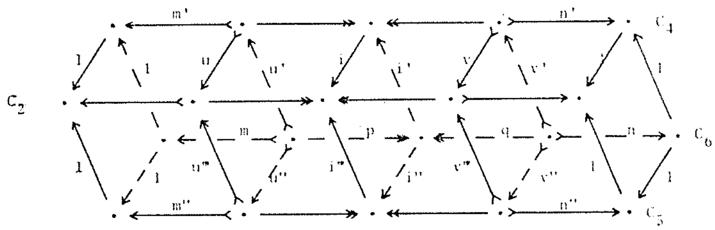
where the upper square is a pushout ($C\theta$ 3*) and also the lower one (obviously) and v is the monic completing the injection (**2.5**). From the diagram follows at once $p' = v = q'$, hence p', q', v are the same isomorphism. Since $q'p = p'q$, we have $p = q$.

5.5. – Proof of **3.1** (necessity of the condition). Let us prove that $\mathscr{C}$ verifies the axioms of $C\theta$-categories.

$C\theta$ 1: obvious. $C\theta$ 2: obvious. $C\theta$ 3: follows from **1.3**. $C\theta$ 3*: follows from **1.4**. $C\theta$ 4: in the diagram **2.1** (1) the upper square, and also the four vertical ones, are bicommutative by **1.2**; therefore, as one can easily prove, also the lower square is bicommutative, hence it is a pullback (1.2). $C\theta$ 4*: proof dual of $C\theta$ 4.

5.6. – Proof of **3.2**. If it exists a diagram of the type **3.2** (1), its four central squares are bicommutative by **1.2**, then $n\tilde{q}p\tilde{m} = n\tilde{q}ii^{-1}p\tilde{m} = nv\tilde{q}_0 p_0 \cdot \tilde{u}\tilde{m} = n_0\tilde{q}_0 p_0\tilde{m}_0$. In the same way $n'\tilde{q}'p'\tilde{m}' = n_0\tilde{q}_0 p_0\tilde{m}_0$. Conversely, let $n\tilde{q}p\tilde{m} = n'\tilde{q}'p'\tilde{m}'$. Let us construct the diagram **3.2** (1) in the following way: u, u' (resp. v, v') are monics completing the pullback of m, m' (resp. n, n') ($C\theta$ 3); $m_0 = mu = m'u'$, $n_0 = nv = n'v'$; i is the isomorphism of $\mathscr{H}$ (hence of $\mathscr{C}$) determined by the two canonical factorizations $(n\tilde{q})(p\tilde{m})$ and $(n'\tilde{q}')(p'\tilde{m}')$ and $j = 1$. At last we find the epic p_0 (q_0); it is sufficient to prove that $i^{-1}pu$ and $p'u'$ are equal epics. We have: $i^{-1}pu = i^{-1}(p\tilde{m})(mu) = (p'\tilde{m}')(m'u') = p'u'$. Let us prove that $p'u'$ is epic in $\mathscr{H}$ (hence in $\mathscr{C}$): $p'u'(p'u')^{\sim} = p'u'\tilde{u}'\tilde{p}' = i^{-1}pu\tilde{u}'\tilde{p}' = i^{-1}p\tilde{m}m'\tilde{p}' = p'\tilde{m}'m'\tilde{p}' = 1$.

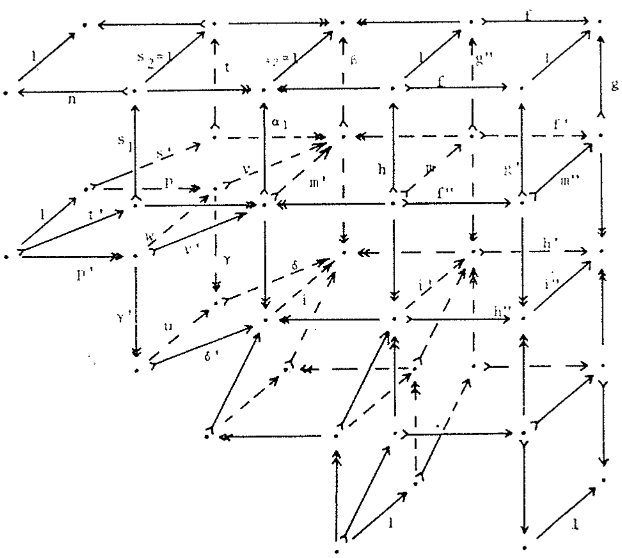
5.7. – Proof that the given relation $\mathscr{R}$ is an equivalence. We can easily verify that the relation between the chains C'' and C given by the *upper portion* of the diagram **3.2** (1) is a preorder, which we denote by $C'' \leqslant C$. So clearly $C\mathcal{R}C' \Leftrightarrow$ it exists C'' such that $C'' \leqslant C$ and $C'' \leqslant C'$. Therefore $\mathcal{R}$ is reflexive and symmetric. Now we prove that $\mathcal{R}$ is transitive.



In fact if $C_1\mathcal{R}C_2$ and $C_2\mathcal{R}C_3$, there exist chains C_4 preceding C_1, C_2 and C_5 preceding C_2, C_3. Then it is sufficient to construct a chain C_6 preceding C_4 and C_5.

First we construct the left and right faces of the diagram (all morphisms are identities) and the three parallel ones as pullbacks; then we define $m = m'u' = m''u''$, $n = n'v' = n''v''$; finally p and q are the epics completing the projections of the faces (u, u', u'', u''') and (v, v', v'', v''') on the face (i, i', i'', i''') (**2.4**), and C_6 is given by the chain (m, p, q, n).

5.8. – Proof that the product is well defined. It is sufficient to prove that, if a is a chain representing α, b and b' are chains representing β with $b \leqslant b'$, and if ba and $b'a$ are any two chains obtained composing b and b' with a, according to the diagram **3.3** (1), then ba and $b'a$ are equivalent, independently on the choices made to construct the squares (4) and (4′). We build the following diagram:

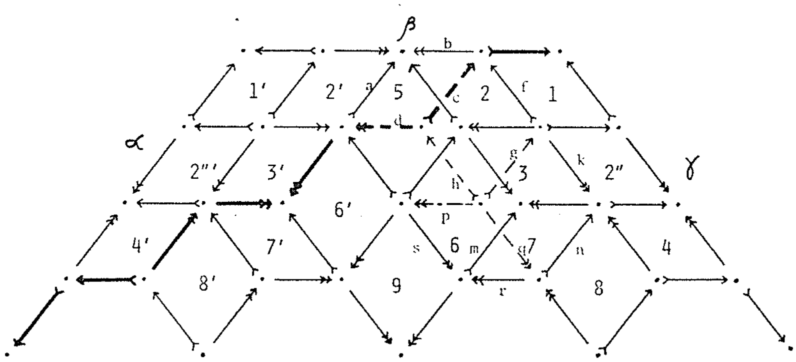


First we write a (the two horizontal upper chains, relied by identities), b and b' (the two vertical right chains, relied by a translation with i'' isomorphism), and the compositions ba, $b'a$ as in **3.3** (1). Then we construct m, m', i, i' such that the diagram is commutative: m exists because the face (f, g, f', g'') is a pullback; m', i' are obtained from **2.3**; i is obtained from **2.5**; m, m', i, i' are all monics. The union of the faces (f, g, f', g'') and (f', m, f'', m'') coincides with the face (f, g', f'', h), which is a pullback, hence also the face (f', m, f'', m'') is a pullback; by $C\theta$ 4, the face (i, i'', h', h'') is a pullback. Since i'' is an isomorphism, also i' and i are isomorphisms.

Now we construct the intermediate chain giving the equivalence between ba and $b'a$. s' and t' are obtained as pullback of $s = s_2 s_1$ and t. vp and $v'p'$ are canonical factorizations. $\delta\gamma$ and $\delta'\gamma'$ are canonical factorizations. w is obtained by **2.3**; it is monic and also epic (being the second factor of an epic) hence w is an isomorphism ($C\theta$ 1); u is obtained analogously and is an isomorphism. The commutative square $(v', \alpha_2\alpha_1, \beta, vw)$ is a pullback because it is the projection of the pullback $(t', s_2 s_1, t, s')$ by epics ($C\theta$ 4). Since $\alpha_2\alpha_1 = \beta m'$ with m' monic, v' must be an isomorphism. Then also δ' and δ are isomorphisms.

This construction is possible even if s_2 and α_2 are only monics (in our case they are identities). So the same construction can be applied to the other half of the diagram. So we obtain a chain (m_1, p_1, q_1, n_1) (where $m_1 = n s_1 t'$, $p_1 = \delta'\gamma' p'$ and q_1, n_1 are obtained analogously) preceding both chains ba and $b'a$, therefore they are equivalent.

5.9. – Proof that the product is associative. Let us consider the following diagram (where numbers denote faces which are not dashed):



The faces 1 and $1'$ are pullbacks; 2, $2'$, $2''$, $2'''$ are commutative ($C\theta$ 1); 3 and $3'$ are pushouts; 4 and $4'$ are commutative ($C\theta$2) with an arbitrary choice of the *lower commuting*; 5 is a pullback; (a, b, c, d) is commutative ($C\theta$ 2) with an arbitrary choice of the commuting. The chain in thick type represents $\beta\alpha$. The face (c, f, g, h) is a pullback; by **2.4** it exists the epic p completing the projection of the pullback (c, f, g, h) on the pullback 5; (g, k, n, q) is commutative ($C\theta$ 1); (p, q, r, s) is a pushout; by **2.5** it exists the monic m completing the injection of the pushout (p, q, r, s) in the pushout 3; m completes the faces 6 and 7; 8 is commutative ($C\theta$ 2).

By a construction symmetric of the preceding one with respect to the central axis of the diagram, we construct the faces 6′, 7′, 8′, then the face **9** as a pushout. In this way the lower chain of the diagram (where we make the possible compositions) represents $\gamma(\beta\alpha)$ and also $(\gamma\beta)\alpha$ and this proves associativity.

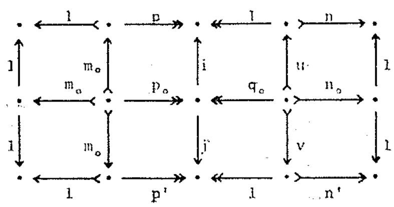
Notice that the chain representing $\gamma\beta\alpha$ can be obtained constructing only the faces which are not dashed (with an arbitrary choice of the *lower commutings*), since they are all bicommutative.

5.10. – Proof that $s: \mathscr{C} \to \mathscr{H}$ is a symmetrization. We must prove: (*a*) s is a functor; (*b*) s satisfies the axioms S_1, S_2, S_3, S_4 of $[4]_1$.

(*a*) It is clear that s maps the identities of $\mathscr{C}$ into the identities of $\mathscr{H}$. If a and b are composable morphisms of $\mathscr{C}$, $a = mp$ and $b = nq$ are their canonical factorizations and $m'q'$ is the canonical factorization of qm, then $ba = (nq)(mp) = (nm')(q'p)$ is the canonical factorization of ba. Hence $s(ba)$ is the morphism represented by the chain $(1, q'p, 1, nm')$ and coincides with $s(b)\,s(a)$, as it is easily verified.

(*b*) Easy to check.

5.11. – Proof that $s: \mathscr{C} \to \mathscr{H}$ verifies the axioms $S\theta$ 1-7. $S\theta$ 1 and $S\theta$ 2 are obvious. Let us prove $S\theta$ 3. If m is a monic of $\mathscr{C}$, it can be easily verified that $\widetilde{s(m)}\,s(m) = 1$, hence $s(m)$ is monic (coretraction); dually, if p is epic $s(p)$ is epic. We prove now that s is faithful. Let a, b be morphisms of $\mathscr{C}$, $a = np$ and $b = n'p'$ their canonical factorizations. If $s(a) = s(b)$ we must have a commutative diagram



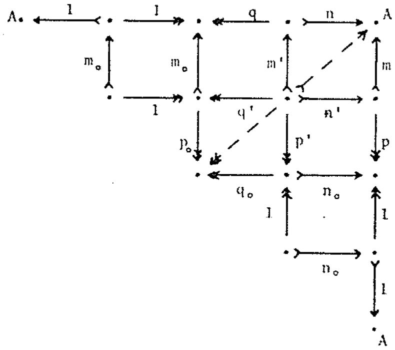
where i, j are isomorphisms and also u, v, q_0 are isomorphisms because they are epics and monics. By **2.6** we have: $(i^{-1}p)m_0 = i^{-1}ip_0 = p_0 = (j^{-1}p')m_0 \Rightarrow$ $\Rightarrow i^{-1}p = j^{-1}p'$, hence

$$a = np = nii^{-1}p = nuq_0^{-1}j^{-1}p' = n_0 q_0^{-1} j^{-1} p' = n' vq_0^{-1} j^{-1} p' = n' jj^{-1} p' = n'p' = b.$$

Now we prove $S\theta$ 6. From now on, we identify $u \in \mathscr{C}$ with $s(u)$. If a morphism α of $\mathscr{H}$ is represented by the chain (m, p, q, n), it can be immediately verified that $\alpha = n\tilde{q}p\tilde{m}$. So $S\theta$ 6 is obvious.

$S\theta$ 4 follows at once from the definition of the product in $\mathscr{H}$.

Now we prove $S\theta$ 5. Since a regular involution category is inverse iff its idempotent morphisms are symmetrical $(\alpha = \tilde{\alpha})$ ([**4**]$_1$, 2.16), let us characterise the idempotent endomorphisms of $\mathscr{H}$. Let $\alpha = (n\tilde{q})(p\tilde{m}) = \mu\pi$ be an idempotent endomorphisms of A. Obviously $\pi\mu = 1_A$, therefore in the diagram giving the product $\pi\mu$.



The lower chain must represent 1_A, hence (**1.1**) m_0, p_0, q_0, n_0 are isomorphisms and $m_0 p_0^{-1} = n_0 q_0^{-1}$. The dashed diagonal, with the isomorphisms $m_0 p_0^{-1} = n_0 q_0^{-1}$ and the monics m' and n', gives the equivalence between the chains $(1, 1, q, n)$ and $(1, 1, p, m)$. Then we have $\pi = \tilde{\mu}$ and $\alpha = \mu\pi = \mu\tilde{\mu}$ is symmetrical. It is clear that a chain representing α is (m, p, p, m) and the idempotent endomorphisms are exactly of this type.

Finally $S\theta$ 7 can be immediately verified by the definition of the product in $\mathscr{H}$.

5.12. – Proof of **3.5** (uniqueness of the $S\theta$-symmetrization). Let $\mathscr{C}$ be a $C\theta$-category, $s: \mathscr{C} \to \mathscr{H}$ a $S\theta$-symmetrization, $s' = s_{\mathscr{C}\theta}: \mathscr{C} \to \mathscr{H}'$ the symmetrization obtained through the symmetrizer θ. To prove that s and s' are isomorphic ([**4**]$_1$, 2.8) it is sufficient to construct the $\sim$-functors $h: \mathscr{H} \to \mathscr{H}'$ and $h': \mathscr{H}' \to \mathscr{H}$ such that $hs = s'$, $h's' = s$.

Construction of h'. The existence of h' follows ([**4**]$_1$, 4.17) from the fact that s maps the squares **3.5** of $\mathscr{C}$ into bicommutative squares of $\mathscr{H}$ (**1.2**).

Construction of h. Let us define $h(n\tilde{q}p\tilde{m}) = s'(n)\widetilde{s'(q)}s'(p)\widetilde{s'(m)}$. We must prove: (1) h is well defined; (2) h is a $\sim$-functor; (3) $hs = s'$.

(1) If $n\tilde{q}p\tilde{m} = n'\tilde{q}'p'\tilde{m}'$, we have the commutative diagram **3.2** (1), where i, j are isomorphisms. Since the squares (u, p, i, p_0) and (i, q, v, q_0) are s'-exact ([**4**]$_1$, 2.22), we have

$$h(n\tilde{q}p\tilde{m}) = s'(n)\widetilde{s'(q)}s'(p)\widetilde{s'(m)} = s'(n)\widetilde{s'(q)}s'(i)s'(i^{-1})s'(p)\widetilde{s'(m)} =$$

$$= s'(n)s'(v)\widetilde{s'(q_0)}s'(p_0)\widetilde{s'(u)}\widetilde{s'(m)} = s'(n_0)\widetilde{s'(q_0)}s'(p_0)\widetilde{s'(m_0)} = h(n_0\tilde{q}_0p_0\tilde{m}_0) ;$$

in the same way

$$h(n'\tilde{q}'p'\tilde{m}') = h(n_0\tilde{q}_0p_0\tilde{m}_0) = h(n\tilde{q}p\tilde{m}) .$$

(2) $h(\beta\alpha) = h(\beta)h(\alpha)$ follows at once from the fact that in the diagram **3.3** (1) all the squares are s'-exact ([**4**]$_1$, 2.22). Finally, $h(1) = 1$ and $h(\tilde{\alpha}) = h(\alpha)$ are obvious.

(3) Let $a = mp$ be the canonical factorization of a morphism of $\mathscr{C}$. We have

$$hs(a) = h(m\tilde{1}p\tilde{1}) = s'(m)\widetilde{s'(1)}s'(p)\widetilde{s'(1)} = s'(mp) = s'(a) .$$

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