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JOHN SCHINAS (*)

On the asymptotic equivalence of linear difference equations. (**)

1. - The notion of the topological equivalence of ordinary differential equations is basic in the modern view of the theory of dynamical systems, since it is the starting point of the structural stability of the flows.

In this paper we are studying another kind of equivalence, the so-called asymptotic equivalence for linear discrete dynamical systems. Strictly speaking such a system is defined as the group homomorphism

$$Z \to \mathrm{Diff}(E): t \to f^t ,$$

where Z is the set of integers and Diff(E) is the space of diffeomorphisms of a Banach space E [4]. Here however, we restrict ourselves to the space of time—dependent linear diffeomorphisms.

The main result of this paper is that two linear discrete dynamical systems are asymptotically equivalent if they are summably comparable and restrictively stable (Definitions 3, 4, 5). This result is the discrete analogue of a problem in [1] (p. 99) for the differential equations, with a proper adaptation to the infinite dimensional linear discrete case. Our basic reference is [3]. (See also [5]).

Denote by: $N(t_0) = \{t_0, t_0 + 1, \dots\}$, where t_0 is any natural number or zero; $I = \{a, a + 1, \dots\}$, where a is a fixed integer; $I(t_0) = \{t_0, t_0 + 1, \dots\}$ $t_0 \in I$; E a Banach space with norm $|\cdot|$; $\boldsymbol{R}^n$ the n-dimensional real Euclidean space with norm: $|x| = \sum_{i=1}^{n} |x_i|$; $L_h(E)$ the space of all linear homeomorphisms from E

(*) Indirizzo: Department of Mathematics, Aristotle University, Thessaloniki, Grecia.

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on E with norm $\|\cdot\|$ induced by $|\cdot|$; $L_h(\boldsymbol{R}^n)$ the space of all $n\times n$ non-singular matrices $A=(a_{ij})$ with norm induce by the norm in $\boldsymbol{R}^n$

$$\|A\|=\max_k\sum_{i=1}^n|a_{ik}|\,;$$

M the identity operator of E.

Our linear discrete dynamical system are introduced by linear difference equations.

Let for each $t\in N(1)$, $A_1(t)\in L_h(E)$ and consider the linear difference equation

$$x_1(t)=A_1(t)\,x_1(t-1)\,, \tag{1}$$

where $x_1(t)\in E$, for $t\in N(0)$.

Definition 1. For any $t_0\in I$, $x_0\in E$, the function $x\colon I(t_0)\to E[y\colon I\diagdown I\cdot(t_0+1)\to E]$ is called *right* [*left*] *solution of* (1), if $x(t_0)=x_0[y(t_0)=x_0]$ and $x(t)$ [resp. $y(t)$] satisfies (1) for all $t\in I(t_0+1)$, $[t\in I(a+1)\diagdown I(t_0+1)]$; if $x(t)$ and $y(t)$ exist and $x(t_0)=y(t_0)=z_0$, the function $z\colon I\to E$, $z(t)=x(t)$, $t\in I(t_0)$, $z(t)=y(t)$, $t\in I\diagdown I(t_0+1)$ will be called *solution of* (1) with $z(t_0)=z_0$.

Obviously the right, [left] solution of (1) exists and is unique, since for all $t\in I(a)$, $A(t)\in L_h(E)$.

In the sequel, juxtaposition of maps denotes composition of maps. Consider the forward solution of the difference operator equation $X_1(t)=A_1(t)\cdot X_1(t-1)$, which is given by

$$X_1(t)=A_1(t)\,A_1(t-1)\ \dots\ A_1(t_0+1)\,M\,,\qquad t\in I(t_0+1)\,. \tag{2}$$

Similarly the backward solution of the same difference operator equation is

$$X_1(t)=A_1^{-1}(t+1)\,A_1^{-1}(t)\ \dots\ A_1^{-1}(t_0)\,M\,,\qquad t\in I(a+1)\diagdown I(t_0+1)\,. \tag{3}$$

Consequently the solution of (1) is given by

$$z_1(t)=X_1(t)\,X_1^{-1}(t_0)\,z_0\,, \tag{4}$$

where $X_1(t)$ is given by (2) for $t\in I(t_0+1)$ and by (3) for $t\in I(a+1)\diagdown I(t_0+1)$.

Definition 2. The difference equation (1) is called *restrictively stable* iff: (i) there exists a $\delta>0$ such that, for any $t_0\in I$ and $|z_0|<\delta$, there exists at least one solution $z(t)$ of (1) with $z(t_0) = z_0$; (ii) for any $\varepsilon > 0$ and $t_0 \in I$, there exists a $\delta(\varepsilon)$, $0 < \delta(\varepsilon) < \delta$ independent of t_0 such that, for any solution $z(t)$ of (1)

$$(5) \qquad |z(t_0)\| < \delta(\varepsilon) \qquad \text{implies} \qquad |z(t)| < \varepsilon .$$

The following lemma can be proved as for the continuous case of the differential equations ([**2**], Ch. II).

Lemma 1. *The difference equation* (1) *is restrictively stable if and only if there exists a constant* $K \geqslant 1$ *such that*

$$\|X_1(t)\| \leqslant K , \qquad \|X_1^{-1}(t)\| \leqslant K , \qquad t \in I .$$

Taking into consideration (4) and Lemma 1, (1) is restrictively stable on $N(O)$ if and only if, there exists a $K \geqslant 1$ such that

$$\sup_{t,s \in N(0)} \|X_1(t) X_1^{-1}(s)\| \leqslant K .$$

Remark 1. From above one can easily conclude that if (1) is restrictively stable then the spectrum $\sigma(A_1)$ of the operator A_1 lies on the unit circle. The inverse is not always true. For instance, if E is a Hilbert space, the spectrum $\sigma(A_1)$ lies on the unit circle if and only if A_1 is similar to a unitary operator [**6**].

In addition to (1) suppose that for all $t \in N(1)$, $A_2(t) \in L_h(E)$ and consider the difference equation

$$(6) \qquad x_2(t) = A_2(t) x_2(t-1) .$$

Definition 3. The difference equations (1) and (2) are called *summably comparable* iff the series of general term $\|A_1(t) - A_2(t)\|$, and the series of general term $\|A_2^{-1}(t) - A_1^{-1}(t)\|$ are both convergent.

Let S_1, S_2 be the sets of the solutions of (1) and (2) respectively.

Definition 4. The difference equations (1) and (6) are called *asymptotically equivalent* iff there exists a one-to-one mapping $F: S_1 \to S_2$ such that $F(x_1)(t)$ tends to $x_1(t)$ as $t \to \infty$.

Note that the notion of asymptotic equivalence is an equivalence relation.

Lemma 2. *If* (1) *and* (6) *are summably comparable and* (1) *is restrictively stable on* $N(0)$, *then* (6) *is also restrictively stable on* $N(0)$.

Proof. Taking the forward solution of the operator equation

$$X_2(t) - A_1(t)\,X_2(t-1) = [A_2(t) - A_1(t)]\,X_2(t-1)\,,$$

we find, for all $t \in I(t_0)$, $t_0 \in N(0)$,

$$X_2(t)\,X_2^{-1}(t_0) = X_1(t)\,X_1^{-1}(t_0) + \sum_{s=t_0+1}^{t} X_1(t)\,X_1^{-1}(s)[A_2(s) - A_1(s)]\,X_2(s-1)\,X_2^{-1}(t_0)\,,$$

and therefore

$$\|X_2(t)\,X_2^{-1}(t_0)\| \leqslant K + \sum_{s=t_0}^{t-1} K\|A_2(s+1) - A_1(s+1)\|\;\|X_2(s)\,X_2^{-1}(t_0)\|\,.$$

Applying the discrete Gronwall's inequality [3]

$$\text{(7)}\qquad \|X_2(t)\,X_2^{-1}(t_0)\| \leqslant K \exp \sum_{s=t_0}^{t-1} K\|A_2(s+1) - A_1(s+1)\| \leqslant K \exp \sum_{s=1}^{\infty} K\|A_2(s) - A_1(s)\|\,.$$

Taking also the backward solution of the operator equation

$$X_2(t-1) - A_1^{-1}(t)\,X_2(t) = [A_2^{-1}(t) - A_1^{-1}(t)]\,X_2(t)\,,$$

we find that, for all $t \in I(a+1) \setminus I(t_0+1)$, $t_0 \in N(O)$

$$X_2(t)\,X_2^{-1}(t_0) = X_1(t)\,X_1^{-1}(t_0) + \sum_{s=t+1}^{t_0} X_1(t)\,X_1^{-1}(s-1)[A_2^{-1}(s) - A_1^{-1}(s)]\,X_2(s)\,X_2^{-1}(t_0)\,,$$

and therefore

$$\|X_2(t)\,X_2^{-1}(t_0)\| \leqslant K + \sum_{s=t+1}^{t_0} K\|A_2^{-1}(s) - A_1^{-1}(s)\|\;\|X_2(s)\,X_2^{-1}(t_0)\|\,.$$

Applying again the discrete Gronwall's inequality,

$$\text{(8)}\qquad \|X_2(t)\,X_2^{-1}(t_0)\| \leqslant K \exp \sum_{s=t+1}^{t_0} K\|A_2^{-1}(s) - A_1^{-1}(s)\| \leqslant K \exp \sum_{s=1}^{\infty} K\|A_2^{-1}(s) - A_1^{-1}(s)\|\,.$$

By combination of (7) and (8) and applying Definition 1 we conclude that (2) is restrictively stable on $N(0)$.

Using Lemma 2 we can prove now the main result.

Theorem. *Suppose that*: i) *equations* (1) *and* (6) *are summably comparable and* ii) *equation* (1) *is restrictively stable on* $N(0)$. *Then*: a) *equations* (1) *and* (6) *are asymptotically equivalent*; b) *the map* $F(\cdot)(0)$ *between the initial values of the corresponding solutions of* (1) *and* (6) *is a homeomorphism*; c) *if* $x_1 \in S_1$, *then*

$$|F(x_1)(t) - x_1(t)| = 0\Big(\sum_{s=t+1}^{\infty} \|A_2(s) - A_1(s)\|\Big) .$$

Proof. *a*) From Lemma 2 and hypotheses i) and ii) it follows that (2) is also restrictively stable on $N(0)$. Let $x_1 \in S_1$ and $X_2(t)$ be the fundamental solution of the operator equation

$$X_2(t) = A_2(t)\, X_2(t-1) , \qquad t \in N(1) .$$

Consider the mapping $F\colon S_1 \to S_2$ which is given by

$$F(x_1)(t) = x_1(t) + \sum_{s=t+1}^{\infty} X_2(t)\, X_2^{-1}(s)[A_1(s) - A_2(s)]\, x_1(s-1) . \tag{9}$$

Since

$$\Big|\sum_{s=t+1}^{\infty} X_2(t)\, X_2^{-1}(s)[A_1(s) - A_2(s)]\, x_2(s-1)\Big| \leqslant$$

$$\leqslant \sup_{s \in N(0)} |x_1(s)| \sup_{t,s \in N(0)} \|X_2(t)\, X_2^{-1}(s)\| \sum_{s=1}^{\infty} \|A_2(s) - A_1(s)\| ,$$

(9) has a meaning. Moreover, if $F(x_1)(t) = x_2(t)$, then $x_2(t)$ is a solution of (6), since

$$F(x_1(t) = x_2(t) = x_1(t) - X_2(t)\, X_2^{-1}(t)[A_1(t) - A_2(t)]\, x_1(t-1) +$$

$$+ \sum_{s=t}^{\infty} X_2(t)\, X_2^{-1}(s)[A_1(s) - A_2(s)]\, x_1(s-1)$$

$$= A_2(t)\, x_1(t-1) + \sum_{s=t}^{\infty} A_2(t)\, X_2(t-1)\, X_2^{-1}(s)[A_1(s) - A_2(s)]\, x_1(s-1)$$

$$= A_2(t)\, F(x_1)(t-1) = A_2(t)\, x_2(t-1) .$$

To prove that the correspondence which has been established by (9) is a one-to-one mapping, we argue as follows. Firstly, let $x_1 \in S_1$. From (9) we get that

$$\lim_{t\to\infty} [F(x_1)(t) - x_1(t)] = 0 . \tag{10}$$

Let $y_2(t) \neq F(x_1)(t)$ be another solut on of (6) such that

$$\lim_{t\to\infty} [y_2(t) - x_1(t)] = 0 . \tag{11}$$

Then, by virtue of (10) and (11), we get

$$\lim_{t\to\infty} [F(x_1)(t) - y_2(t)] = 0 .$$

For any fixed $t \in I(t_0 + 1)$ and $\tau \in I(a + 1) \setminus I(t_0 + 1)$, consider the left solutions $x_2(\tau)$ and $y_2(\tau)$ which are given, according to (4), by

$$x_2(\tau) = X_2(\tau)\, X_2^{-1}(t)\, x_2(t) , \qquad y_2(\tau) = X_2(\tau)\, X_2^{-1}(t)\, y_2(t) .$$

From these expressions and because (1) and (6) are restrictively stable

$$|y_2(t) - x_2(t)| \leqslant |y_2(t) - x_2(t)| \max_{\tau \in I(a+1) \setminus I(t_0+1)} \|X_2(t)\, X_2^{-1}(t)\| = K\, |y_2(t) - x_2(t)\| .$$

If now $t \to \infty$ in the last inequality, we get $y_2(t) = x_2(t)$ for all $t \in N(1)$. So, given $x_1 \in S_1$, we find a unique $x_2 \in S_2$ which corresponds to x_1.

Conversely, because both equations (1) and (6) are restrictively stable, given any $x_2 \in S_2$, we can define, by virtue of

$$G(x_2)(t) = x_2(t) + \sum_{s=t+1}^{\infty} X_1(t)\, X_1^{-1}(s)[A_2(s) - A_1(s)]\, x_2(s-1) , \tag{12}$$

a unique $x_1 \in S_1$ and so (1) and (6) are asymptotically equivalent and $G(x)(t) = F^{-1}(x)(t)$, for all $x \in S_2$.

b) Setting $t = 0$ in (9) and (12) we find, for all $x_1, x_2 \in E$,

$$F(x_1) = F(x_1)(0) = \Big[M + \sum_{s=1}^{\infty} X_2^{-1}(s)[A_1(s) - A_2(s)]\, X_1(s-1)\Big]\, x_1 ,$$

$$F(x_2) = F(x_2)(0) = \Big[M + \sum_{s=1}^{\infty} X_1^{-1}(s)[A_2(s) - A_1(s)]\, X_2(s-1)\Big]\, x_2 .$$

Consequently, $FG = M$ according to the proof of the case *a*).

c) To establish *c*) we use (9), which implies that

$$|F(x_1)(t) - x_1(t)| \leqslant K \sum_{s=t+1}^{\infty} \|A_1(s) - A_2(s)\| .$$

Example. Consider the system of difference equations in $E = \boldsymbol{R}^2$, $t \in N(1)$,

$$x_1(t) = A_1(t)\, x_1(t-1) = \left(1 + \frac{1}{2^t}\right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x_1(t-1) , \tag{13}$$

$$x_2(t) = A_2(t)\, x_2(t-1) = \left(1 - \frac{1}{2^t}\right) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} x_2(t-1) . \tag{14}$$

Then, since the series

$$\sum_{t=1}^{\infty} \|A_1(t) - A_2(t)\| = \sum_{t=1}^{\infty} \frac{1}{2^{t-1}} , \qquad \sum_{t=1}^{\infty} \|A_1^{-1}(t) - A_2^{-1}(t)\| = \sum_{t=1}^{\infty} \frac{2 \cdot 2^t}{2^{2t} - 1} .$$

are convergent, and $\|X_1(t)\| < e^2$, $\|X_1^{-1}(t)\| \leqslant \frac{2}{3}$ (which implies that (13) is restrictively stable), then also (14) is restrictively stable. In fact, $\|X_2(t)\| \leqslant e^{-2}$, $\|X_2^{-1}(t)\| \leqslant e^4$ for all $t \in N(1)$. Moreover, after some easy calculation we find that the one-to-one mapping between them is the following

$$F(x_1)(t) = x_1(t) + \sum_{s=t+1}^{\infty} \frac{1}{2^{s-1}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \left[\prod_{k=1}^{s-t} \left(1 - \frac{1}{2^{k+t}}\right) \right] x_1(s-1) .$$

Finally the linear homeomorphism $F \in L_h(\boldsymbol{R}^2)$ of the initial values of the corresponding solutions is given by

$$F(x) = \left\{ M + \sum_{s=1}^{\infty} \frac{1}{2^{s-1}} \left(1 - \frac{1}{2^s}\right)^{-1} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1} \left[\prod_{k=1}^{s} \left(1 - \frac{1}{2^{2k}}\right) \right] \right\} x .$$

References.

[1] E. Coddington and N. Levinson, *Theory of ordinary differential equations*, McGraw-Hill, New York 1955.

[2] W. A. Coppel, *Stability and asymptotic behavior of differential equations*, Heath Mathematical Monographs, London 1965.

[3] Ju. L. Daleckii and M. G. Krein, *Stability of solutions of differential equations in Banach space*, Amer. Math. Soc. Transl. (1974).

[4] J. W. Robbin, *Topological conjugacy and structural stability for discrete dynamical systems*. Bull. Amer. Math. Soc. **78** (1972), 923-952.

[5] J. Schinas, *Stability and conditional stability of time-dependent difference equations in Banach spaces*, J. Inst. Math. Appl. **14** (1974), 335-346.

[6] A. E. Taylor, *Introduction to functional analysis*, J. Wiley, London 1958.

Sommario.

Si studia l'equivalenza asintotica tra sistemi dinamici lineari discreti. Si prova che due tali sistemi che sono sommabilmente confrontabili e restrittivamente stabili sono asintoticamente equivalenti.

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