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PREM CHANDRA (*)

On the $|E, q|$ summability of a Fourier series and its conjugate series. (**)

1. - Definitions and notations.

Let $\sum a_n$ be a given infinite series ([1]). Then the series $\sum a_n$ is said to be *absolutely summable* (E, q) $(q > 0)$ or symbolically $\sum a_n \in |E, q|$ $(q > 0)$, if

$$\sum (q+1)^{-n} \left| \sum_{k=0}^{n} \binom{n}{k} q^{n-k} a_k \right|$$

is convergent. Also see Chandra [1].

We define the summability $|E, 0|$ equivalent to the absolute convergence.

Let $f \in L(-\pi, \pi)$ and be periodic with period 2π, and let

$$(1.1) \qquad f(t) \sim \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} A_n(t) , \qquad \text{say.}$$

The conjugate series of (1.1), at $t = x$, is

$$(1.2) \qquad \sum_{n=1}^{\infty} (b_n \cos nx - a_n \sin nx) = \sum_{n=1}^{\infty} B_n(x) .$$

We assume throughout $A_0(x) = a_0$ and $B_0(x) = 0$.

We use the following notations in this paper. Let r be a non-negative integer.

(*) Indirizzo: School of Studies in Mathematics, Vikram University, Ujjain (M.P.) India.

(**) Ricevuto: 24-III-1975.

([1]) Summations are over $0, 1, 2, \dots, \infty$ when there is no indication to the contrary.

(1.3) $$\varphi(t) = \tfrac{1}{2}\{f(x+t) + f(x-t)\},$$

(1.4) $$\psi(t) = \tfrac{1}{2}\{f(x+t) - f(x-t)\},$$

(1.5) $$\varphi_\alpha(t) = \frac{\alpha}{t^\alpha}\int_0^t (t-u)^{\alpha-1}\,\varphi(u)\,dy \qquad (\alpha > 0).$$

Similarly we define $\psi_\alpha(t)$ for $\alpha > 0$.

(1.6) $$(F(nt))_r = \left[\left(\frac{\partial}{\partial y}\right)^r (F(ny))\right]_{y=t},$$

(1.7) $$(F(nt))_{-r} = [(\textstyle\iiint \ldots r\text{-times})\, F(ny)(dy)^r]_{y=t}.$$

2. - Introduction.

Concerning the $|E, q|$ $(0 < q < 1)$ summability of Fourier series and conjugate series, the following theorems, due to Mohanty and Mohaptra [**3**], are known

Theorem A. Let $0 < p < 1$ and $0 < q < 1$. Then $\varphi(t)\log 1/t \in BV(0, p)$ implies that $\sum A_n(x) \in |E, q|$.

Theorem B. Let $0 < p < 1$ and $0 < q < 1$. Then $\psi(t)\log 1/t \in BV(0, p)$ and $\psi(t)t^{-1} \in L(0, p)$ imply that $\sum B_n(x) \in |E, q|$.

In what follows, we prove the following theorems.

Theorem 1. *Let α be a positive integer. Then $t^{-\alpha}\varphi_\alpha(t) \in BV(0, \pi)$ implies that $\sum A_n(x) \in |E, q|$ $(0 < q < 1)$.*

Theorem 2. *Let α be a positive integer. Then $t^{-\alpha}\psi_\alpha(t) \in BV(0, \pi)$ implies that $\sum B_n(x) \in |E, q|$ $(0 < q < 1)$.*

In view of a known result (Bosanquet [**2**]): $\varphi_\alpha(t) \in BV(0, \pi)$ implies that $\sum A_n(x) \in |C, \beta|$ $(\beta > \alpha \geqslant 0)$, it may appear that the condition of Theorem 1 is somewhat artificial. Therefore, in section **7** of this paper, we show that the conditions imposed upon the generating functions of the Fourier series and its conjugate series, in the above theorems, are not un-natural.

In section **8**, we replace the set of conditions, imposed upon the generating functions of the Fourier series and conjugate series in Theorems 1 and 2, by another set of conditions, by showing their equivalence.

3. – We require the following order-estimates for the proof of the theorems

$$(1 + q)^{-n}(1 + q^2 + 2q\cos t)^{\frac{1}{2}n} = O\{\exp(-nqt^2/2\pi^2)\} \qquad (0 < t \leqslant \pi/2), \tag{3.1}$$

$$\sum_{m=0}^{n} \binom{n}{m} q^{n-m} m^k \exp(imt) \sim n^k (1 + q^2 + 2q\cos t)^{\frac{1}{2}(n-k)} \exp\{i(kt + (n-k)\theta)\}, \tag{3.2}$$

where k is a non-negative integer and

$$\theta = \tan^{-1}\left\{\frac{\sin t}{q + \cos t}\right\} \qquad (0 < t \leqslant \pi).$$

For the proof of (3.1), see Ray [**4**], lemma 2.

Proof of (3.2). We write

$$A = \sum_{m=0}^{n} \binom{n}{m} q^{n-m} m^k \exp(imt).$$

Now, since

$$\binom{n}{m} = \frac{n(n-1)\dots(n-k+1)}{m(m-1)\dots(m-k+1)} \binom{n-k}{m-k} \qquad (k \leqslant m),$$

we have

$$A \sim n^k q^n \sum_{m=0}^{n} \binom{n-k}{m-k} \frac{\exp(imt)}{q^m}.$$

As

$$\sum_{m=0}^{n} \binom{n-k}{m-k} \frac{\exp(imt)}{q^m} = \sum_{m=k}^{n} \binom{n-k}{m-k} \frac{\exp(imt)}{q^m} = \sum_{m=0}^{n-k} \binom{n-k}{m} \frac{\exp(i(m+k)t)}{q^{m+k}}$$

$$= \frac{\exp(ikt)}{q^k} \left(1 + \frac{\exp(it)}{q}\right)^{n-k}$$

$$= q^{-n} \exp(ikt)(q + \cos t + i\sin t)^{n-k}$$

$$= q^{-n}(1 + q^2 + 2q\cos t)^{\frac{1}{2}(n-k)} \exp\{i(kt + (n-k)\theta)\},$$

where $\theta = \tan^{-1}\{\sin t/(q + \cos t)\}$, we get a proof for (3.2).

4. – We require the following lemmas.

Lemma 1. *The summability method* $|E, q|$ $(q > 0)$ *includes* $|E, 0|$.

Lemma 2. *Let s be a non-negative integer. Then, uniformly in* $0 < t \leqslant \pi$,

$$\sum (q+1)^{-n} \left| \sum_{m=0}^{n} \binom{n}{m} q^{n-m} \frac{\sin mt}{(m+1)^{s+1}} \right| = O\left(\log \frac{2\pi}{t}\right); \tag{4.1}$$

and

$$\sum (q+1)^{-n} \left| \sum_{m=0}^{n} \binom{n}{m} q^{n-m} \frac{\cos mt}{(m+1)^{s+1}} \right| = O\left(\log \frac{2\pi}{t}\right). \tag{4.2}$$

Proof. First consider the case $s > 0$. The series $\sum \sin nt/(n+1)^{s+1}$ is both absolutely and uniformly convergent in $0 < t \leqslant \pi$. Therefore, by Lemma 1, we have

$$\sum (q+1)^{-n} \left| \sum_{m=0}^{n} \binom{n}{m} q^{n-m} \frac{\sin mt}{(m+1)^{s+1}} \right| = O(1),$$

uniformly in $0 < t \leqslant \pi$.

The case $s = 0$ of (4.1) for $0 < t \leqslant \delta < 1$, is due to Chandra ([**1**], lemma 3) and for $0 < t \leqslant \pi$, it can be proved similarly.

The proof of (4.2) is similar to that of (4.1).

5. - Proof of Theorem 1.

We have

$$A_n(x) = \frac{2}{\pi} \int_0^{\pi} \varphi(u) \cos nu \, du .$$

Integrating, α-times, by parts, we have

$$A_n(x) = \frac{2}{\pi} \sum_{s=0}^{\alpha-1} \frac{(-1)^s}{\Gamma(s+1)} \pi^{s+1} \varphi_{s+1}(\pi) (\cos n\pi)_s + \\ + \frac{2}{\pi} \frac{(-1)^\alpha}{\Gamma(\alpha+1)} \int_0^{\pi} u^{-\alpha} \varphi_\alpha(u) u^{2\alpha} (\cos nu)_\alpha \, du .$$

Now, since $u^{-\alpha}\varphi_\alpha(u) \in BV(0, \pi)$, we can write $u^{-\alpha}\varphi_\alpha(u) = g_1(u) - g_2(u)$, where $g_i(u)$ $(i = 1, 2)$ are positive, monotonic increasing and bounded in $0 \leqslant u \leqslant \pi$. Therefore, by the second mean value theorem, we have $(0 \leqslant t_1 \leqslant \pi)$

$$\int_0^\pi u^{-\alpha}\varphi_\alpha(u)\, u^{2\alpha}(\cos nu)_\alpha\, du = A_0 \int_0^\pi u^{2\alpha}(\cos nu)_\alpha\, du + A_1 \int_0^{t_1} u^{2\alpha}(\cos nu)_\alpha\, du + \\ + A_2 \int_0^{t_2} u^{2\alpha}(\cos nu)_\alpha\, du \qquad (0 \leqslant t_2 \leqslant \pi),$$

where the constants A_i $(i = 0, 1, 2)$ are defined below

$$A_0 = g_1(\pi) - g_2(\pi) \qquad \text{and} \qquad A_i = (-1)^{i-1}\{g_i(0) - g_i(\pi)\} \qquad (i = 1, 2).$$

Thus

$$A_n(x) = \frac{2}{\pi}\sum_{s=0}^{\alpha-1} \frac{(-1)^s}{\Gamma(s+1)}\, \pi^{s+1}\, \varphi_{s+1}(\pi)(\cos n\pi)_s + \frac{2}{\pi}\frac{(-1)^\alpha}{\Gamma(\alpha+1)} \sum_{i=0}^{2} A_i \int_0^{t_i} u^{2\alpha}(\cos nu)_\alpha\, du,$$

where $t_0 = \pi$ and t_i $(i = 1, 2)$ are some numbers such that $0 \leqslant t_i \leqslant \pi$.

Integrating, 2α-times, by parts, we have

$$\int_0^{t_i} u^{2\alpha}(\cos nu)_\alpha\, du = \sum_{s=0}^{\alpha-1}(-1)^s(\cos nt_i)_{\alpha-s-1}\,(t_i^{2\alpha})_s + \\ + \sum_{s=0}^{\alpha-1}(-1)^{s+\alpha}(\cos nt_i)_{-(1+s)}(t_i^{2\alpha})_{s+\alpha} \int_0^{t_i} + (u^{2\alpha})_{2\alpha}\,(\cos nu)_{-\alpha}\, du \\ = \sum_{s=0}^{\alpha-1} K(\cos nt_i)_s\,(t_i)^{s+1+\alpha} + \\ + \sum_{s=0}^{\alpha-1} K(t_i)^{\alpha-s}\, n^{-(1+s)}\, S(nt_i) + \int_0^{t_i} (u^{2\alpha})_{2\alpha}(\cos nu)_{-\alpha}\, du,$$

where K's denote the constants depending upon s and α, and not necessarily the same at each occurrence; and $S(nt_i)$ is $\cos nt_i$ or $\sin nt_i$ according as s is an odd or even integer. Therefore, collecting the results, we obtain

$$A_n(x) = \frac{2}{\pi}\sum_{s=0}^{\alpha-1} \frac{(-1)^s}{\Gamma(s+1)}\, \pi^{s+1}\, \varphi_{s+1}(\pi)(\cos n\pi)_s + \\ + \frac{2}{\pi}\frac{(-1)^\alpha}{\Gamma(\alpha+1)} \sum_{i=0}^{2} A_i \sum_{s=0}^{\alpha-1} K(\cos nt_i)_s (t_i)^{s+1+\alpha} + \\ + \frac{2}{\pi}\frac{(-1)^{\alpha})}{\Gamma(\alpha+1)} \sum_{i=0}^{2} A_i \sum_{s=0}^{\alpha-1} K(t_i)^{\alpha-s}\, n^{-(1+s)}\, S(nt_i) + \\ + \frac{2}{\pi}\frac{(-1)^\alpha}{\Gamma(\alpha+1)} \sum_{i=0}^{2} A_i \int_0^{t_i} (u^{2\alpha})_{2\alpha}(\cos nu)_{-\alpha}\, du$$

$$(5.1) \qquad A_n(x) = \sum_{r=1}^{4} P_n^{(r)}, \qquad \text{say}.$$

Now, since $P_n^{(4)} = O\{1/(n+1)^{1+\alpha}\}$, the series $\sum P_n^{(4)} \in |E, q|$ $(q > 0)$, by Lemma 1. Also the series $\sum P_n^{(1)} \in |E, q|$ $(0 < q < 1)$ whenever the series $\sum P_n^{(2)} \in |E, q|$ $(0 < q < 1)$. Therefore, for the proof of Theorem 1, it will be sufficient to show, uniformly in $0 < t \leqslant \pi$, that

$$\sum (q+1)^{-n} \left| \sum_{m=0}^{n} \binom{n}{m} q^{n-m} \frac{S(mt)}{(m+1)^{s+1}} \right| = O(t^{s-\alpha}) \tag{5.2}$$

and

$$\sum (q+1)^{-n} \left| \sum_{m=0}^{n} \binom{n}{m} q^{n-m} (\cos mt)_s \right| = O(t^{-1-s-\alpha}) , \tag{5.3}$$

where integers s and α are such that $0 \leqslant s \leqslant \alpha - 1$.

The proof of (5.2) follows from Lemma 2.

Proof of (5.3). By (3.2), we have

$$\sum (q+1)^{-n} \left| \sum \binom{n}{m} q^{n-m} (\cos mt)_s \right| \leqslant \sum (q+1)^{-n} (n+1)^s (1 + q^2 + 2q \cos t)^{\frac{1}{2}(n-s)}$$

$$= \Sigma , \quad \text{say.}$$

Thus, for the proof of (5.3), it is enough to show that $\sum = O(t^{-1-s-\alpha})$ $(0 \leqslant s \leqslant \alpha - 1)$, uniformly in $0 < t \leqslant \pi/2$.

Now, by using (3.1) and writing d for $qt^2/2\pi^2$, we have

$$\Sigma = O\left\{\sum (n+1)^s \exp(-nd)\right\} = O\left\{\sum \binom{n+s}{n} (\exp(-d))^n\right\} ;$$

since $\dfrac{(n+1)^s}{\Gamma(s+1)} \sim \dbinom{n+s}{n}$,

$$\Sigma = O\{(1 - \exp(-d))^{-s-1}\} = O(d^{-s-1}) ;$$

since $e^d/(e^d - 1) = O(d^{-1})$, for $d > 0$,

$$\Sigma = O(t^{-2(s+1)}) = O(t^{-1-s-\alpha}) ,$$

uniformly in $0 < t \leqslant \pi/2$, where $0 \leqslant s \leqslant \alpha - 1$.

This terminates the proof of Theorem 1.

6. - Proof of Theorem 2.

We have

$$B_n(x) = \frac{2}{\pi}\int_0^\pi \psi(t)\sin nt\,dt\,.$$

Proceeding as in Theorem 1 of this paper, the series $\sum B_n(x) \in |E, q|$ $(0<q<1)$, if the following inequalities, for integers s and α such that $0\leqslant s\leqslant \alpha-1$ and uniformly in $0<t\leqslant\pi$, hold

$$\sum (q+1)^{-n}\left|\sum_{m=0}^{n}\binom{n}{m} q^{n-m}\int_0^t (u^{2\alpha})_{2\alpha}\,(\sin mu)_{-\alpha}\,du\right| < \infty, \tag{6.1}$$

$$\sum (q+1)^{-n}\left|\sum_{m=0}^{n}\binom{n}{m} q^{n-m}(\sin mt)_s\right| = O(t^{-1-s-\alpha}), \tag{6.2}$$

and

$$\sum (q+1)^{-n}\left|\sum_{m=0}^{n}\binom{n}{m} q^{n-m}\frac{\bar{S}(mt)}{(m+1)^{s+1}}\right| = O(t^{s-\alpha})\,, \tag{6.3}$$

where $\bar{S}(mt)$ is $\sin mt$ or $\cos mt$ according as s is an odd or even integer.

The proof of (6.1) follows from Lemma 1, since

$$\int_0^\pi (u^{2\alpha})_{2\alpha}\,(\sin mu)_{-\alpha}\,du = O\{(m+1)^{-1-\alpha}\}\,,$$

uniformly in $0<t\leqslant\pi$. And the proof of (6.3) follows from Lemma 2.

By using (3.2) and arguing as in the proof of (5.3), a proof for (6.2) may be worked out.

This completes the proof of Theorem 2.

7. - In this section we prove the following theorems.

Theorem 3. *Let $\delta>0$ and α be a positive integer. Then $t^{-(\alpha-\delta)}\varphi_\alpha(t)\in BV(0,\pi)$ is not a sufficient condition for $|E, q|^{(0<q<1)}$ summability of Fourier series at a point $t=x$.*

Theorem 4. *Let $\delta>0$ and α be a positive integer. Then $t^{-(\alpha-\delta)}\psi_\alpha(t)\in BV(0,\pi)$ is not a sufficient condition for $|E, q|$ $(0<q<1)$ summability of the conjugate series of a Fourier series, at a point $t=x$.*

7.1. We shall require the following lemma for the proof of above theorems.

Lemma 3. *Let s and α be integers such that $0 \leqslant s \leqslant \alpha - 1$ and let $\delta > 0$. Then*

$$t^{1+s+\alpha-\delta} \sum (q+1)^{-n} \left| \sum_{m=0}^{n} \binom{n}{m} q^{n-m} \cdot \begin{Bmatrix} (\cos mt)_s \\ (\sin mt)_s \end{Bmatrix} \right| \to \infty ,$$

as $t \to +0$.

Proof. Let $1 > \delta > 0$ without loss of generality and let

$$P = \sum (q+1)^{-n} \left| \sum_{m=0}^{n} \binom{n}{m} q^{n-m} (\cos mt)_s \right| .$$

Now, by (3.2), we have

$$P = \sum (q+1)^{-n} (n+1)^s (1 + q^2 + 2q \cos t)^{\frac{1}{2}(n-s)} |S(n, s, t, \theta)| ,$$

where $\theta = \tan^{-1}\{\sin t/(q + \cos t)\}$ and $S(n, s, t, \theta)$ is $\cos\{st + (n-s)\theta\}$ or $\sin\{st + (n-s)\theta\}$ according as s is an even or an odd integer such that $0 \leqslant s \leqslant \alpha - 1$. Now, further, we have

$$Q = \Gamma(s+1) \sum \binom{n+s}{n} (q+1)^{-n} (1 + q^2 + 2q \cos t)^{\frac{1}{2}(n-s)} |S(n, s, t, \theta)|$$

$$\geqslant \frac{\Gamma(s+1)}{(1+q)^s} \sum \binom{n+s}{n} \left[1 - \left(\frac{2q^{\frac{1}{2}} \sin \frac{1}{2} t}{1+q} \right)^2 \right]^{\frac{1}{2}(n-s)} S^2(n, s, t, \theta)$$

$$\geqslant \frac{\Gamma(s+1)}{2(1+q)^s (\cos \tau)^s} \sum \binom{n+s}{n} (\cos \tau)^n [1 + (-1)^s \cos\{2st + 2(n-s)\theta\}] ;$$

since $2q^{\frac{1}{2}} \sin \frac{1}{2} t = (1+q) \sin \tau$,

$$Q = C \sum \binom{n+s}{n} (\cos \tau)^n + (-1)^s C \sum \binom{n+s}{n} (\cos \tau)^n \cos\{2st + 2(n-s)\theta\}$$

$$= C\Sigma_1 + (-1)^s C \Sigma_2 ,$$

where $C = (\Gamma(s+1))/(2(1+q)^s (\cos \tau)^s)$.

Now, it may be observed that

$$\Sigma_1 = (1 - \cos\tau)^{-s-1} = \frac{2^{-s-1}}{(\sin\frac{1}{2}\tau)^{2(s+1)}}$$

and

$$\Sigma_2 = \text{Real part of } \left\{\exp(i2s(t-\theta)) \sum \binom{n+s}{n} (\exp(i2\theta)\cos\tau)^n\right\}$$

$$= \text{Real part of } \left\{\exp(i2s(t-\theta))(1-\exp(i2\theta)\cos\tau)^{-s-1}\right\}$$

$$= R^{-s-1}\cos\left\{(s+1)\varphi + 2s(t-\theta)\right\},$$

where

$$\tan\varphi = \frac{\sin 2\theta\cos\tau}{1-\cos 2\theta\cos\tau},$$

and

$$R^2 = (1-\cos 2\theta\cos\tau)^2 = \sin^2 2\theta\cos^2\tau$$

$$= (1-\cos\tau)^2 + 2\cos\tau(1-\cos 2\theta) = (2\sin^2\tfrac{1}{2}\tau)^2 + 4\cos\tau\sin^2\theta.$$

We further observe that

$$\lim_{t\to+0} t^{2(s+1)}\Sigma_1 = \left\{\frac{2(1+q)^2}{q}\right\}^{s+1} \tag{7.1.1}$$

and

$$\lim_{t\to+0} t^{s+1}\Sigma_2 = \left(\frac{1+q}{q^{\frac{1}{2}}}\right)^{s+1}\left\{\frac{4}{q}\right\}^{-\frac{1}{2}(s+1)}. \tag{7.1.2}$$

Therefore we have, by (7.1.2), $t^{1+s+\alpha-\delta}\Sigma_2 = (t^{1+s}\Sigma_2)t^{\alpha-\delta} \xrightarrow[t\to+0]{} 0$, and, by (7.1.1), $t^{1+s+\alpha-\delta}\Sigma_1 = (t^{2(1+s)}\Sigma_1)t^{-1-s+\alpha-\delta} \xrightarrow[t\to+0]{} \infty$, and hence it follows that: $t^{1+s+\alpha-\delta}Q \to \infty$, as $t \to +0$, which implies that $t^{1+s+\alpha-\delta}P \to \infty$, as $t \to +0$.

Similarly it may be shown that $t^{1+s+\alpha-\delta}T \to \infty$, as $t \to +0$, where

$$T = \sum (q+1)^{-n}\left|\sum_{m=0}^{n}\binom{n}{m}q^{n-m}(\sin mt)_s\right|.$$

This completes the proof of the lemma.

7.2. Proof of Theorem 3. Without any loss of generality, we take $1 > > \delta > 0$ for the proof of the theorem.

From Theorem 1, we have

$$A_n(x) = \frac{2}{\pi} \sum_{s=0}^{\alpha-1} \frac{(-1)^s}{\Gamma(s+1)} \pi^{s+1} \varphi_{s+1}(\pi)(\cos n\pi)_s + \\ + \frac{2}{\pi} \frac{(-1)^\alpha}{\Gamma(\alpha+1)} \int_0^\pi u^{-\alpha+\delta} \varphi_\alpha(u)\, u^{2\alpha-\delta} (\cos nu)_\alpha \, du \, .$$

Writing $u^{\delta-\alpha}\varphi_\alpha(u) = J_1(u) - J_2(u)$, where $J_i(u)$ $(i = 1, 2)$ are positive, monotonic increasing and bounded in $0 \leqslant u \leqslant \pi$, and proceeding as in Theorem 1, we get

$$A_n(x) = \frac{2}{\pi} \sum_{s=0}^{\alpha-1} \frac{(-1)^s}{\Gamma(s+1)} \pi^{s+1} \varphi_{s+1}(\pi)(\cos n\pi)_s +$$

$$+ \frac{2}{\pi} \frac{(-1)^\alpha}{\Gamma(\alpha+1)} \sum_{i=0}^{2} B_i \sum_{s=0}^{\alpha-1} K(\cos nt_i)_s (t_i)^{s+1+\alpha-\delta} +$$

$$+ \frac{2}{\pi} \frac{(-1)^\alpha}{\Gamma(\alpha+1)} \sum_{i=0}^{2} B_i \sum_{s=0}^{\alpha-1} K(t_i)^{-s+\alpha-\delta} n^{-(1+s)} S(nt_i) +$$

$$+ \frac{2}{\pi} \frac{(-1)^\alpha}{\Gamma(\alpha+1)} \sum_{i=0}^{2} B_i \int_0^{t_i} (u^{2\alpha-\delta})_{2\alpha} (\cos nu)_{-\alpha} \, du \, ,$$

$$A_n(x) = b_n + c_n + d_n + e_n \, ,$$

where $B_0 = J_1(\pi) - J_2(\pi)$, $B_i = (-1)^{i-1}\{J_i(0) - J_i(\pi)\}$ $(i = 1, 2)$; $t_0 = \pi$, t_i $(i = = 1, 2)$ are some numbers such that $0 \leqslant t_i \leqslant \pi$; K's denote the constants depending upon s and α, not necessarily the same at each occurrence; and $S(nt_i)$ is $\cos nt_i$ or $\sin nt_i$ according as s is an odd or even integer. Thus from the above relation, we have

$$c_n = A_n(x) - b_n - d_n - e_n \, . \tag{7.2.1}$$

Now, the series $\sum c_n \in |E, q|$ $(0 < q < 1)$, if and only if

$$\Sigma_0 = \sum (q+1)^{-n} \left| \sum_{m=0}^{n} \binom{n}{m} q^{n-m} c_m \right| < \infty \, .$$

But, by (7.2.1), we have

$$\Sigma_0 \leqslant \sum (q+1)^{-n} \left| \sum_{m=0}^{n} \binom{n}{m} q^{n-m} A_m(x) \right| + \sum (q+1)^{-n} \left| \sum_{m=0}^{n} \binom{n}{m} q^{n-m} b_m \right| +$$

$$+ \sum (q+1)^{-n} \left| \sum_{m=0}^{n} \binom{n}{m} q^{n-m} d_m \right| + \sum (q+1)^{-n} \left| \sum_{m=0}^{n} \binom{n}{m} q^{n-m} e_m \right| ,$$

(7.2.2) $$\Sigma_0 = \Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4 .$$

Since the series $\sum |e_m| = O\{\sum 1/(m+1)^{1+\alpha-\delta}\} < \infty$, $\Sigma^4 < \infty$ follows from Lemma 1. And the boundedness of Σ_3 follows from Lemma 2 and that of Σ_2 follows from (5.3), case $t = \pi$. Therefore for

(7.2.3) $$\Sigma_1 < \infty ,$$

i.e. $\sum A_n(x) \in |E, q|$ $(0 < q < 1)$, it is necessary, from (7.2.2), that

(7.2.4) $$\Sigma_0 < \infty ,$$

which is true if and only if

$$\Sigma_5 = t^{1+s+\alpha-\delta} \sum (q+1)^{-n} \left| \sum_{m=0}^{n} \binom{n}{m} q^{n-m} (\cos mt)_s \right| = O(1) ,$$

uniformly in $0 < t \leqslant \pi$, where integers s and α are such that $0 \leqslant s \leqslant \alpha - 1$ But, by Lemma 3, $\Sigma_5 \to \infty$, as $t \to +0$, therefore Σ_5 is not bounded uniformly in $0 < t \leqslant \pi$, which implies that (7.2.4) does not hold and hence (7.2.3) does not hold, that is $\sum A_n(x)$ is not summable $|E, q|$ $(0 < q < 1)$.

This terminates the proof of Theorem 3.

7.3. Proof of Theorem 4. – Its proof runs parallel to that of Theorem 3.

Remark 1. Proceeding as in Theorem 3, it can be shown that the condition $\left(t^{-\alpha}/g(k/t)\right)\varphi_\alpha(t) \in BV(0, \pi)$ is not a sufficient condition for $|E, q|$ $(0 < q < 1)$ summability of a Fourier series, at a point $t = x$, where $g(k/t)$ is a function of the type $(\log k/t)^c$, $(\log_2 k/t)^c$, ..., $(\log_b k/t)^c$, where k is suitable positive constant such that $g(k/\pi) > 0$, $c > 0$, $\log_1 = \log$ and $\log_b = \log \log_{b-1}$. A similar remark is also valid for the conjugate series.

8. – Let $F(t) \in L(0, a)$, where $a > 0$, and let

$$P(t) = F(t) - \frac{1}{t} \int_0^t F(u)\, du . \tag{8.1}$$

Then we prove the following lemma which shall be used in this section.

Lemma 4. *Let* $c > 0$ *and let* $F(t) \in L(0, a)$ $(a > 0)$. *Then*

$$t^{-c} F(t) \in BV(0, a) \tag{8.2}$$

is and only if

$$\text{(i)}\quad F(+0) = 0 , \qquad \text{(ii)}\quad t^{-c} P(t) \in BV(0, a) . \tag{8.3}$$

Proof. We first prove that (8.3) implies (8.2). Let $\varepsilon > 0$. Then, on substituting the value of $P(t)$ from (8.1), we have

$$P(t) + \int_\varepsilon^t \frac{P(u)}{u}\, du = F(t) - \frac{1}{t} \int_0^t F(u)\, du + \int_\varepsilon^t \frac{1}{u} \left\{ F(u) - \frac{1}{u} \int_0^u F(y)\, dy \right\} du ,$$

$$P(t) + \int_\varepsilon^t \frac{P(u)}{u}\, du = F(t) - \frac{1}{\varepsilon} \int_0^\varepsilon F(u)\, du , \tag{8.4}$$

after some straightforward manipulation. Now, since $F(+0) = 0$, by (8.3) (i), we follow that $(1/\varepsilon) \int_0^\varepsilon F(u)\, du \to 0$, as $\varepsilon \to 0$. And hence, on taking the limit $\varepsilon \to 0$ in (8.4), we get

$$F(t) = P(t) + \int_0^t \frac{P(u)}{u}\, du , \tag{8.5}$$

which is the inverse transformation of (8.1), under (8.3) (i). Now suppose that $t^{-c} P(t) \in BV(0, a)$. Then we can write $t^{-c} P(t) = P_1(t) - P_2(t)$, where $P_1(t)$, $P_2(t)$ are non-negative and non-decreasing in $0 \leqslant t \leqslant a$. Thus, from (8.5), we get

$$t^{-c} F(t) = t^{-c} P(t) + t^{-c} \int_0^t u^{c-1} \{P_1(u) - P_2(u)\}\, du .$$

And, by using the transformation $u = tv$ in the integral of the above equation, we have

$$t^{-c}F(t) = t^{-c}P(t) + \int_0^1 v^{c-1}P_1(tv)\,dv - \int_0^1 v^{c-1}P_2(tv)\,dv .$$

Now the integrals

$$\int_0^1 v^{c-1}P_1(tv)\,dv , \qquad \int_0^1 v^{c-1}P_2(tv)\,dv$$

are non-negative and non-decreasing functions of t in $0 \leqslant t \leqslant a$; hence their difference is a function of bounded variation over $(0, a)$. And, since $t^{-c}P(t) \in BV(0, a)$, we follow, from the above equation, that $t^{-c}F(t) \in BV(0, a)$.

The converse implication, i.e. (8.2) implies (8.3), may be proved in a similar way from (8.1).

Now, we prove the following theorem

Theorem 5. *Let α be a positive integer. Then*

$$\varphi_\alpha(+0) = 0 \qquad \textit{and} \qquad t^{-\alpha}P_\alpha(t) \in BV(0, \pi) \tag{8.6}$$

imply that $\sum A_n(x) \in |E, q|$ $(0 < q < 1)$, *where*

$$P_\alpha(t) = \varphi_\alpha(t) - \frac{1}{t}\int_0^t \varphi_\alpha(u)\,du . \tag{8.7}$$

Remark 2. It may be observed that $\varphi_\alpha(t) \in BV(0, \pi)$ implies that $P_\alpha(t) \in BV(0, \pi)$, but converse in not necessarily true. For example, let $\varphi_\alpha(t) = \log \pi/t$, which is not of bounded variation over $(0, \pi)$, but $P_\alpha(t) = 1 \in BV(0, \pi)$. Therefore, alone $P_\alpha(t) \in BV(0, \pi)$ is lighter condition than $\varphi_\alpha(t) \in BV(0, \pi)$.

Proof of Theorem 5. On replacing $P(t)$ and $F(t)$, respectively, by $P_\alpha(t)$ and $\varphi_\alpha(t)$ in (8.1) and Lemma 4, we obtain, respectively, (8.7) and, on letting $c = \alpha$ and $a = \pi$ in Lemma 4, (8.6) implies that $t^{-\alpha}\varphi_\alpha(t) \in BV(0, \pi)$. And therefore the proof of the theorem follows from Theorem 1.

A result, corresponding to Theorem 5 for the conjugate series, also holds.

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