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On the absolute Nörlund summability factors. (**)

1. – Let S_n denote the nth partial sum of a given infinite series $\sum a_n$. Let $\{p_n\}$ be a sequence of constants, real or complex and let us write

$$P_n = \sum_{\nu=0}^{n} p_\nu , \qquad P_{-1} = p_{-1} = 0 .$$

The

$$(1.1) \qquad T_n = \frac{1}{P_n} \sum_{\nu=0}^{n} p_{n-\nu} S_\nu = \frac{1}{P_n} \sum_{\nu=0}^{n} P_\nu a_{n-\nu} \qquad (P_n \neq 0)$$

defines the sequence $\{T_n\}$ of Nörlund means of the sequence $\{S_n\}$ generated by the coefficients $\{p_n\}$ [**7**].

The series $\sum a_n$ is said to be absolute summable (N, p_n) with index k or summable $|N, p_n|_k$ $(k > 0)$ if

$$(1.2) \qquad \sum n^{k-1} | T_{n+1} - T_n |^k < \infty ,$$

when $K = 1$; this definition reduces to the customary definition of absolute Nörlund summability, as given by Mears [**5**].

In the special case in which

$$(1.3) \qquad p_n = \binom{n + \alpha - 1}{\alpha - 1} = \frac{\Gamma(n + \alpha)}{\Gamma(n + 1)\,\Gamma(\alpha)} \qquad (\alpha > 0) ,$$

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the Nörlund mean reduces to (c, α) mean [**3**]. Thus the summability $|N, p_n|$, where p_n is defined by (1.3) is the same as $|c, \alpha|$. Again, when

$$(1.4) \qquad p_n = \frac{1}{n+1}, \quad P_n \sim \log n \text{ as } n \to \infty,$$

the Nörlund mean reduces to the harmonic mean [**9**].

The conditions for the regularity of the method of summability (N, p_n) defined by (1.1) are

$$(1.5) \qquad \lim_{n \to \infty} \frac{p_n}{P_n} = 0$$

and

$$(1.6) \qquad \sum_{\nu=0}^{n} |p_\nu| = O(P_n), \quad \text{as } n \to \infty .$$

If p_n is real and non-negative, (1.6) is automatically satisfied, and then (1.5) is the necessary and sufficient condition for the regularity of the method.

2. – Let $f(t)$ be a periodic function with period 2π and integrable (L) over $(-\pi, \pi)$. Let its Fourier series be

$$(2.1) \qquad \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + p_n \sin nt) \equiv \tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} A_n(t) .$$

We write $\varphi(t) = \{f(x+t) + f(x-t)\}/2$.

A sequence $\{u_n\}$ is said to be convex [**11**] if

$$\Delta^2 u_n \geqslant 0 , \quad (n = 1, 2, 3, \ldots) ,$$

where $\Delta u_n = u_n - u_{n+1}$ and $\Delta^2 u_n = \Delta(\Delta u_n)$.

3. – Given a sequence $\{\lambda_n\}$ if the series $\sum a_n \lambda_n$ is absolutely summable in some sense, while in general $\sum a_n$ is itself not so summable, then $\{\lambda_n\}$ is said to be the absolute summability factor of the series $\sum a_n$.

Kogbetliantz has proved the following theorem [4] on summability factors for absolute Cesàro summability.

Theorem A. If a series $\sum a_n$ is $|c, \alpha|$ summable, then the series $\sum a_n \varepsilon_n$ is summable $|c, \beta|$ for $\beta \leqslant \alpha$, $\alpha, \beta > 0$, if $\varepsilon_n = 1/(n+1)^{\alpha-\beta}$.

In 1952, Peyerimhoff [8] gave a similar proof of the above theorem.

Further in 1965 N. Kishore [6] established a similar theorem for the case of Nörlund summability when the series in summable $|c, 1|$. His theorem runs as given below:

Theorem B. If a series $\sum a_n$ is $|c, 1|$ summable and if $\{p_n\}$ be a non-increasing sequence of real and nonnegative numbers, then the series $\sum (a_n P_n)/n$ is $|N, p_n|$ summable, where $P_n = \sum_{\nu=0}^{n} p_\nu$.

The object of this paper is to extend the above Theorem B for $|N, p_n|_k$ summability. In fact we prove:

Theorem. *If the series $\sum a_n$ is $|c, 1|_k$ summable and if $\{p_n\}$ be a non-increasing sequence of real and nonnegative numbers, then the series $\sum (a_n P_n)/n$ is summable $|N, p_n|_k$ where $P_n = \sum_{\nu=0}^{n} p_\nu$.*

4. - Proof of the Theorem. Since the case $k = 1$ of our theorem is due to N. Kishore [6]. We prove for $k > 1$ only.

Further, since the series $\sum a_n$ is summable $|c, 1|_k$ $(k \geqslant 1)$, we have

$$\sum n^{-1} |\tau_n^1|^k < \infty , \tag{4.1}$$

where τ_n^1 denotes the nth Cesàro mean of order one of the sequence $\{na_n\}$.

Let T_n denotes the Nörlund mean of the series $\sum (a_n P_n/n) = \sum u_n$; then we have to show that

$$\sum n^{k-1} |T_{n+1} - T_n|^k < \infty . \tag{4.2}$$

Now

$$T_n = \frac{1}{P_n} \sum_{\nu=1}^{n} p_{n-\nu} S_\nu = \frac{1}{P_n} \sum_{\nu=1}^{n} P_{n-\nu} u_\nu .$$

Since $P_{-1} = 0$

$$T_{n+1} - T_n = \sum_{\nu=1}^{n+1} \left(\frac{P_{n+1-\nu}}{P_{n+1}} - \frac{P_{n-\nu}}{P_n} \right) u_\nu = \sum_{\nu=1}^{n+1} \nu a_\nu \left(\frac{P_{n+1-\nu}}{P_{n+1}} - \frac{P_{n-\nu}}{P_n} \right) \frac{P_\nu}{\nu^2} .$$

Applying Abel's transformation and denoting $t_n = \sum_{\nu=1}^{n} \nu a_\nu$, $t_0 = 0$ and $\Delta\lambda_n = \lambda_n - \lambda_{n+1}$, we have

$$(4.3)\qquad |T_{n+1} - T_n| = \left| \sum_{\nu=1}^{n} t_\nu \Delta \left\{ \left(\frac{P_{n+1-\nu}}{P_{n+1}} - \frac{P_{n-\nu}}{P_n} \right) \frac{P_\nu}{\nu^2} \right\} + P_0 \frac{t_{n+1}}{(n+1)^2} \right|$$

$$\leqslant \left| \sum_{\nu=1}^{n} t_\nu \left(\frac{P_{n+1-\nu}}{P_{n+1}} - \frac{P_{n-\nu}}{P_n} \right) \Delta \frac{P_\nu}{\nu^2} \right| +$$

$$+ \left| \sum_{\nu=1}^{n} t_\nu \frac{P_{\nu+1}}{(\nu+1)^2} \Delta \left(\frac{P_{n+1-\nu}}{P_{n+1}} - \frac{P_{n-\nu}}{P_n} \right) \right\} + \frac{P_0 |t_{n+1}|}{(n+1)^2}$$

$$\leqslant |L_n^{(1)}| + |L_n^{(2)}| + |L_3^{(3)}|, \quad \text{say}.$$

By Minkowski's inequality, it is therefore sufficient to prove that

$$(4.4)\qquad \sum n^{k-1} |L_n^{(1)}|^k < \infty ,$$

$$(4.5)\qquad \sum n^{k-1} |L_n^{(2)}|^k < \infty ,$$

and

$$(4.6)\qquad \sum n^{k-1} |L_n^{(3)}|^k < \infty .$$

Proof of (4.4). Since $\{p_n\}$ is a non-negative, non-increasing sequence, it is easy to see that $(P_{n+1-\nu}/P_{n-\nu}) \geqslant (P_{n+1}/P_n)$ for all $\nu \leqslant n$, and hence [1]

$$\sum_{n=1}^{m} n^{k-1} |L_n^{(1)}|^k = \sum_{n=1}^{m} n^{k-1} \left\{ \left| \sum_{\nu=1}^{n} t_\nu \left(\frac{P_{n+1-\nu}}{P_{n+1}} - \frac{P_{n-\nu}}{P_n} \right) \Delta \frac{P_\nu}{\nu^2} \right) \right| \right\}^k$$

$$= \sum_{n=1}^{m} n^{k-1} \left\{ \sum_{\nu=1}^{n} |t_\nu^k| \left(\frac{P_{n+1-\nu}}{P_{n+1}} - \frac{P_{n-\nu}}{P_n} \right) \Delta \left(\frac{P_\nu}{\nu^2} \right) \right\} \times$$

$$\times \left\{ \left| \sum_{\nu=1}^{n} \left(\frac{P_{n+1-\nu}}{P_{n+1}} - \frac{P_{n-\nu}}{P_n} \right) \Delta \left(\frac{P_\nu}{\nu^2} \right) \right| \right\}^{k-1}$$

$$\leqslant A \sum_{n=1}^{m} n^{k-1} \left\{ \sum_{\nu=1}^{n} |t_\nu^k| \left(\frac{P_{n+1-\nu}}{P_{n+1}} - \frac{P_{n-\nu}}{P_n} \right) \Delta \left(\frac{P_\nu}{\nu^2} \right) \right\} \cdot \left\{ \frac{1}{n^2} \right\}^{k-1}$$

$$\leqslant A \sum_{\nu=1}^{m} \left| t_\nu^k \Delta \left(\frac{P_\nu}{\nu^2} \right) \right| \sum_{n=\nu}^{m} \left(\frac{P_{n+1-\nu}}{P_{n+1}} - \frac{P_{n-\nu}}{P_n} \right) \times \frac{1}{n^{k-1}}$$

(1) A is a constant not necessarily the same at each occurrence.

$$\leqslant A \sum_{\nu=1}^{m} \frac{|t_\nu^k \Delta(P_\nu/\nu^2|)}{\nu^{k-1}} \sum_{n=\nu}^{m} \left(\frac{P_{n+1-\nu}}{P_{n+1}} - \frac{P_{n-\nu}}{P_n}\right) \leqslant A \sum_{\nu=1}^{m} \frac{|t_\nu^k \Delta(P_\nu/\nu^2)|}{\nu^{k-1}} \cdot \frac{P_{m+1-\nu}}{P_{m+1}}$$

$$\leqslant A \left[\sum_{\nu=1}^{m} \frac{|t_k^k|}{\nu^{k-1}} \left(-\frac{P_{\nu+1}}{\nu^2} + P_{\nu+1} \cdot \Delta\left(\frac{1}{\nu^2}\right)\right)\right]$$

$$\leqslant A \left[\sum_{\nu=1}^{m} \frac{p_{\nu+1}|t_\nu|^k}{\nu^{k+1}}\right] + A \left[\sum_{\nu=1}^{m} \frac{P_{\nu+1}|t_\nu|^k}{\nu^{k+2}}\right].$$

Now, since $\sum a_n$ is $|c, 1|_k$ summable $\sum(|t_\nu|^k/\nu^{k+1})$ is convergent; thus since $P_{\nu+1} \leqslant (\nu+1)p_0$. We have

$$\sum n^{k-1} |L_n^{(1)}|^k = O\left\{\sum_{\nu=1}^{m} \frac{|t_\nu|^k}{\nu^{k+1}}\right\} + O\left\{\sum_{\nu=1}^{m} \frac{|t_\nu|^k}{\nu^{k+1}}\right\},$$

(4.7) $$\sum n^{k-1} |L_n^{(1)}|^k = O(1), \quad \text{as } m \to \infty.$$

P r o o f o f (4.5).

$$\sum_{n=1}^{m} n^{k-1} |L_n^{(2)}|^k = \sum_{n=1}^{m} n^{k-1} \left\{\left|\sum_{\nu=1}^{n} t_\nu \frac{P_{\nu+1}}{(\nu+1)^2} \left(\frac{p_{n+1-\nu}}{P_{n+1}} - \frac{p_{n-\nu}}{p_n}\right)\right|\right\}^k$$

$$\leqslant A \sum_{n=1}^{m} n^{k-1} \left\{\sum_{\nu=1}^{n} |t_\nu|^k \frac{P_{\nu+1}}{(\nu+1)^2} \left(\frac{p_{n+1-\nu}}{P_{n+1}} - \frac{p_{n-\nu}}{P_n}\right)\right\} \times$$

$$\times \left\{\sum_{\nu=1}^{n} \frac{P_{\nu+1}}{(\nu+1)^2} \left(\frac{p_{n+1-\nu}}{P_{n+1}} - \frac{p_{n-\nu}}{P_n}\right)\right\}^{k-1}$$

$$\leqslant A \sum_{n=1}^{m} n^{k-1} \sum_{\nu=1}^{n} |t_\nu|^k \frac{P_{\nu+1}}{(\nu+1)^2} \left(\frac{p_{n+1-\nu}}{P_{n+1}} - \frac{p_{n-\nu}}{P_n}\right) \times \{\sum_{\nu=1}^{n} 1/\nu^2\}^{k-1}$$

$$\leqslant A \sum_{\nu=1}^{m} |t_\nu|^k \frac{P_{\nu+1}}{(\nu+1)^2} \sum_{n=\nu}^{m} \left(\frac{p_{n-\nu}}{P_n} - \frac{p_{n+1-\nu}}{P_{n+1}}\right)$$

$$\leqslant A \sum_{\nu=1}^{m} \frac{|t_\nu|^k}{(\nu+1)^2} \cdot \frac{P_{\nu+1} p_0}{P_\nu} \leqslant A \sum_{\nu=1}^{m} \frac{|t_\nu|^k}{\nu^2},$$

whence as $m \to +\infty$

(4 8) $$\sum_{n=1}^{m} n^{k-1} |L_n^{(2)}|^k = O(1).$$

Proof of (4.6).

$$\sum_{n=1}^{m} n^{k-1} |L_n^{(3)}|^k \leqslant A \sum n^{k-1} \left\{ \frac{|t_{n+1} P_0|}{(n+1)^2} \right\}^k$$

$$\leqslant A \sum n^{k-1} \left\{ \frac{|t_{n+1}|}{(n+1)^2} \right\}^k \leqslant A \sum \frac{|t_{n+1}|^k}{(n+1)^{k+1}},$$

whence as $m \to +\infty$

$$(4.9) \qquad \sum_{n=1}^{m} n^{k-1} |L_n^{(3)}|^k = O(1) .$$

Hence, $\sum n^{k-1} |T_{n+1} - T_n|^k < \infty$, which proves the theorem.

5. – It may be remarked here that if $k = 1$ and $P_n \sim \log n$ the following theorem of Singh [**10**] becomes the corollary of our theorem.

Theorem. *In the series* $\sum a_n$ *is summable* $|c, 1|$, *then the series* $\sum a_n \log (n+1)/n$ *is summable* $|N, 1/n+1|$.

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