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A quasilinear equation of hyperbolic type arising from neutron transport. (**)

1. - Introduction.

The mathematical structure of monoenergetic neutron transport initial-value problems under various boundary conditions has been studied in several papers (see, for instance, $[\mathbf{3}]_1$, $[\mathbf{3}]_2$, [**16**]).

Recently [**8**] Hendry considered a neutron transport problem in a slab with moving boundaries; he obtained a formal solution but he did not attempt an existence proof. The existence of a unique solution has been discussed in [**4**]. The original and practical interest in the problem with moving boundaries arose in studies of nuclear reactor accidents [**7**].

The approach to the problem is based on some results for abstract evolution equations in the Hilbert space L^2 and, more generally, in L^p spaces for $p>1$ [**15**]. However, from a physical point of view, the L^1 space is more appropriate than L^p spaces with $p>1$ $[\mathbf{13}]_2$, [**14**], (see also section **2**).

In this paper, we shall study a more general monoenergetic neutron trasport initial-value problem (i.e., with moving boundaries and with cross sections

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depending on the temperature). Specifically, we consider the following integro-differential system

$$(1.1)\quad \begin{cases} \dfrac{\partial N}{\partial t} = v\mu \dfrac{\partial N}{\partial x} - v\,\Sigma(x, t, T)\,N + \dfrac{v}{2}\,\gamma(x, t, T)\displaystyle\int_{-1}^{1} N(x, \mu', t)\,d\mu' , \\ \dfrac{\partial T}{\partial t} = -h[T(x, t) - T_f] + k(x, t, T)\,b(t)\displaystyle\int_{-a_1}^{a_1} dx' \int_{-1}^{1} N(x', \mu', t)\,d\mu' , \\ 0 < t \leqslant t' , \qquad |x| \leqslant a_1 , \qquad |\mu| < 1 , \qquad b(t) = \dfrac{a(t)}{a_1} , \end{cases}$$

with boundary conditions

$$(1.2)\qquad N(-a_1, \mu, t) = 0 \quad \mu \in (0, 1) , \qquad N(+a_1, \mu, t) = 0 \quad \mu \in (-1, 0)$$

and initial conditions

$$(1.3)\qquad N(x, \mu, 0) = N_0(x, \mu) , \qquad T(x, 0) = T_0(x) , \qquad |x| \leqslant a_1 .$$

In (1.1)-(1.3), $N = N(x, \mu, t)$ is the neutron density function, $T = T(x, t)$ is the temperature at x and t, $a(t)$ is a continuously differentiable function such that $0 < a(0) \leqslant a(t) \leqslant a_1$ for any $t \in [0, t']$, and we use standard symbols [**2**], [**5**], [**13**]$_1$, [**13**]$_2$. Following [**4**], we assume that

$$(1.4)\qquad \Sigma(x, t, T) = \gamma(x, t, T) = k(x, t, T) = 0$$

for $x \in [-a_1, a(t)]$ and $x \in [a(t), a_1]$.

If we suppose that the freed-back due to a variation of temperature is linear, we have [**13**]$_1$, [**13**]$_2$

$$(1.5)\qquad \Sigma(x, t, T) = \Sigma_0(x, t)(1 + \alpha_1\theta) , \qquad \gamma(x, t, T) = \gamma_0(x, t)(1 + \alpha_2\theta) ,$$

$$(1.6)\qquad k(x, t, T) = k_0(x, t)(1 + \alpha_3\theta) , \qquad T(x, t) = \theta(x, t) + T_r ,$$

where $\Sigma_0(x, t)$, $\gamma_0(x, t)$ and $k_0(x, t)$ satisfy (1.4), $\alpha_i \in R_1$ for $i = 1, 2, 3$, and T_r is a suitable reference temperature.

Moreover, we make the following assumption:

a.1) $0 < S_i(x, t) < \bar{S}_i$, $|x| < a(t)$, $t \in [0, t']$,

$S_i(x, t) = 0$, $x \in [-a_1, a(t)]$, $x \in [a(t), a_1]$;

a.2) for any $t \in [0, t']$, $S_i(x, t)$ is continuous with respect to x, a.e. on $[-a_1, a_1]$;

a.3) $S_i(x, t)$ is continuously differentiable with respect to t, uniformly in x, a.e. on $[-a_1, a_1]$;

a.4) for any $t \in [0, t']$, $\partial S_i/\partial t$ is continuous and bounded with respect to x, a.e. on $[-a_1, a_1]$, with $|\partial S_i/\partial t| < S_i^*$;

where the symbol $S_i(x, t)$, $i = 1, 2, 3$, denotes $\Sigma_0(x, t)$, $\gamma_0(x, t)$ and $k_0(x, t)$ respectively. Due to relations (1.5)-(1.6), system (1.1) can be put into the form

$$
(1.7)\quad \begin{cases} \dfrac{\partial N}{\partial t} = A_1 N + [v\gamma_0(x, t)J - v\Sigma_0(x, t)I]N - v\theta[\alpha_2\gamma_0(x, t)J - \alpha_1\Sigma_0(x, t)I]N \\[2ex] \dfrac{\partial \theta}{\partial t} = -h\theta + k_0(x, t)b(t)B_1 N + \alpha_3 k_0(x, t)b(t)\theta B_1 N + h(T_f - T_r), \end{cases}
$$

where the operators A_1, J and B_1 are defined in section **2**, and where I is the identity operator.

2. - The mathematical setting.

In order to formulate the system (1.7) together with the boundary and initial conditions (1.2)-(1.3) as an abstract evolution problem, we put $Q_1 = [-a_1, a_1]$, $Q_2 = [-1, 1]$ and $Q = Q_1 \times Q_2$. We denote by X_1 the Banach space $L_1(Q)$ of all complex valued functions $f_1(x, \mu)$ such that $|f_1(x, \mu)|$ is Lebesgue integrable in Q and by X_2 the Banach space $C[Q_1]$ of continuous functions $f_2(x)$. The norms in X_1 and in X_2 are defined respectively by

$$
\|f_1\|_1 = \int_{-a_1}^{a_1} dx \int_{-1}^{1} |f_1(x, \mu)|\,d\mu, \qquad \|f_2\|_2 = \max\{|f_2(x)|, x \in Q_1\}, \quad f_1 \in X_1, \quad f_2 \in X_2.
$$

Then $X = X_1 \times X_2$, with norm

$$
(2.1)\qquad \|f\| = \bar{k}_0\|f_1\|_1 + h\|f_2\|_2, \qquad f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \in X,
$$

is a Banach space. We remark that the use of X_1 is justified by the fact, that $\|N\|_1$ is the total number of neutrons in the slab at the instant t. The constants $\bar{k}_0$ and h in the norm (2.1) are used to adjust dimensions, so that $\|\cdot\|$ is proportional to an energy per unit time.

Finally, let $Y = C([0, t'], X)$ the Banach space of continuous functions $u\colon [0, t'] \to X$ equipped with norm $|||u||| = \sup\{\|u(t)\|, t \in [0, t']\}$.

We now define the following linear operators:

$$(2.2)\qquad A_1 f_1 = -v\mu\frac{\partial f_1}{\partial x}, \quad D(A_1) = \{f_1 \in X_1 : A_1 f_1 \in X_1;\ f_1 \text{ satisfies } (1.2)\}, \quad R(A_1) \subset X_1,$$

$$(2.3)\qquad Jf_1 = \frac{1}{2}\int_{-1}^{1} f_1(x,\mu)\,d\mu, \qquad D(J) = X_1, \qquad R(J) \subset X_1,$$

$$(2.4)\qquad B_1 f_1 = \int_{-a_1}^{a_1} dx \int_{-1}^{1} f_1(x,\mu)\,d\mu; \qquad D(B_1) = X_1, \qquad R(B_1) \subset X_2,$$

$$(2.5)\qquad B_0(t) f_1 = v\gamma_0(x,t) Jf_1 - v\Sigma_0(x,t) f_1, \quad D(B_0(t)) = X_1, \qquad R(B_0(t)) \subset X_1,$$

$$(2.6)\qquad B_1(t) f_1 = k_0(x,t)\, b(t)\, B_1 f_1, \qquad D(B_1(t)) = X_1, \qquad R(B_1(t)) \subset X_2,$$

$$(2.7)\qquad Af = \begin{bmatrix} A_1 & 0 \\ 0 & -hI \end{bmatrix}\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \qquad D(A) = D(A_1) \times X_2, \quad R(A) \subset X,$$

$$(2.8)\qquad B(t) f = \begin{bmatrix} B_0(t) & 0 \\ B_1(t) & 0 \end{bmatrix}\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \qquad D(B(t)) = X, \qquad R(B(t)) \subset X,$$

and nonlinear operator

$$(2.9)\qquad F(t, f) = \begin{bmatrix} v f_2 \alpha_2 \gamma_0(x,t) Jf_1 - v\alpha_1 f_2 \Sigma_0(x,t) If_1 \\ \alpha_3 f_2 B_1(t) f_1 \end{bmatrix},$$

$$D(F(t,f)) = [0, t'] \times X, \quad R(F(t,f)) \subset X.$$

3. - The abstract problem.

Due to (2.2)-(2.9) the system (1.7) with boundary and initial conditions (1.2)-(1.3) is transformed into the abstract semilinear initial value problem

$$(3.1)\qquad \frac{du}{dt} = Au(t) + B(t)u(t) + F(t,u) + v_0, \qquad \lim_{t \to 0^+} u(t) = u_0,$$

where $u(t)$ is a transformation from $[0, t']$ into X, the limit and differentiability are defined in the strong sense, and where

$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} N(x, \mu, t) \\ \theta(x, t) \end{bmatrix}, \quad u_0 = \begin{bmatrix} N_0(x, \mu) \\ T_0(x) - T_r \end{bmatrix}, \quad v_0 = \begin{bmatrix} 0 \\ T_f - T_r \end{bmatrix}.$$

In Lemma 4.1 of the section **4**, we shall prove that $A \in G(1, 0)$ that is A is the generator of a strongly continuous semigroup of contraction operators [**1**], [**10**]$_1$, [**11**]; therefore, the semilinear evolution equation (3.1) is of hyperbolic type and it can be formally reduced to the nonlinear integral equation [**10**]$_2$

$$u(t) = w_0(t) + \int_0^t \exp[(t-s)A]\{B(s)u(s) + F(s,u(s))\}\,ds\,, \tag{3.2}$$

where

$$w_0(t) = \exp[tA]u_0 + \int_0^t \exp[(t-s)A]v_0\,ds\,.$$

In the mathematical theory of semilinear equations of evolution of hyperbolic type we have the following results [**10**]$_2$.

If the nonlinear term $G(t, u) = B(t)u(t) + F(t, u(t))$ is continuous on $[0, t'] \times X$ to X and uniformly Lipschitz continuous in u, in a neighborhood of $u_0 \in X$, then for any $u_0 \in X$, (3.2) has a unique solution $u \in Y$. Following Browder [**6**], a such solution is said a « mild solution » of (3.1). Moreover, if $G(t, u)$ has continuous derivatives G_t and G_u, where G_u is the Fréchet derivative [**11**], then the mild solution is a « strict solution » (i.e. $u(t)$ is continuous on $[0, t']$, continuously differentiable on $(0, t']$ and $Au(t)$ exists on $[0, t']$).

Lemmas 4.1-4.6 of section **4** lead to the following Theorem

Theorem 3.1. *System* (1.7) *with boundary and initial conditions* (1.2)-(1.3) *has a unique* (*local in t*) *mild solution $u \in Y$ which is also a strict solution.*

4. - Properties of the operators.

We now obtain some basic properties of the operators defined in section **2**, which prove the Theorem 3.1.

Lemma 4.1. *A is densely defined and dissipative operator with $R(I-A) = X$* (i.e. $A \in G(1, 0)$ [**12**]).

Proof. From [15], [16] and since $-hI \in B(X_2)$, it follows that $D(A)$ is dense in X. Following [17], we have that the equation

$$(zI - A)f = g \qquad g \in X\,,\ z > 0\,,$$

has a unique solution $f \in D(A)$, that is $R(zI - A) = X$ for any $z > 0$.

Moreover, we have

$$\|R(z, A)\,g\| = \|f\| \leqslant z^{-1}\,\|g\|\ ;$$

hence, if $g = (I - sA)f$ with $f \in D(A)$, we obtain

$$\|f\| = \left\| \frac{1}{s} R\left(\frac{1}{s}\,, A\right) g \right\| \leqslant \|g\| = \|(l - sA)f\|\ ,$$

which proves that A is an dissipative operator.

The following lemmas can easily proved

Lemma 4.2. *$J \in B(X_1)$ and $\|Jf_1\|_1 \leqslant \|f_1\|_1$; $B_1 \in B(X_1, X_2)$ and $\|B_1 f_1\|_2 \leqslant \leqslant \|f_1\|_1$.*

Lemma 4.3. i) *$B_0(t)$ is a continuously differentiable mapping* (c.d.m.) *from $[0, t']$ to $B(X_1)$ with derivative (in the sense of the norm in $B(X_1)$) given by*

$$\frac{\mathrm{d}}{\mathrm{d}t} B_0(t) = v\left(\frac{\partial}{\partial t}\gamma_0(x, t)\,J - \frac{\partial}{\partial t}\,\Sigma_0(x, t) I\right) \in B(X_1)\ .$$

ii) *$B_1(t)$ is a* c.d.m. *from $[0, t']$ to $B(X_1, X_2)$ with derivative (in the sense of the norm in $B(X_1, X_2)$) given by*

$$\frac{\mathrm{d}}{\mathrm{d}t}\, B_1(t) = \left(b(t)\,\frac{\mathrm{d}}{\mathrm{d}t}\,k_0(x, t) - b(t)\,k_0(x, t)\right) B_1 \in B(X_1, X_2)\ .$$

iii) *$B(t)$ is a* c.d.m. *from $[0, t']$ to $B(X)$ with derivative in the sense of the norm in $B(X)$ given by*

$$\frac{\mathrm{d}}{\mathrm{d}t}\, B(t) = \begin{bmatrix} \dfrac{\mathrm{d}}{\mathrm{d}t}\, B_0(t) & 0 \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\, B_1(t) & 0 \end{bmatrix} \in B(X)\ .$$

Proof. i) From assumption a.2) and $f_1 \in X_1$ it follows that integrals

$$\int_{-a_1}^{a_1} |S_i(x,t)| \left[\int_{-1}^{1} |f_1(x,\mu)| \, d\mu \right] dx \qquad (i = 1, 2)$$

exist. Thus, from assumption *a*.1) we obtain

$$\|B_0(t) f_1\|_1 \leqslant v(\bar{\gamma}_0 + \bar{\Sigma}_0) \|f_1\|_1 . \tag{4.1}$$

(4.1) with assumption a.3) proves that $B_0(t)$ is a c.m. from $[0, t']$ to $B(X_1)$.
Moreover, given $\varepsilon > 0$ and $t \in [0, t']$, we obtain from assumption a.3) and Lemma 4.2

$$\left\| \frac{B_0(t+\Delta t) - B_0(t)}{\Delta t} f_1 - v\left(\frac{\partial}{\partial t}\gamma_0(x,t) J - \frac{\partial}{\partial t}\Sigma_0(x,t) I\right) f_1 \right\| < \varepsilon \|f_1\|_1 ,$$

since

$$v\left| \frac{\gamma_0(x, t+\Delta t) - \gamma_0(x,t)}{\Delta t} - \frac{\partial}{\partial t}\gamma_0(x,t) \right| + v \left| \frac{\Sigma_0(x, t+\Delta t) - \Sigma_0(x,t)}{\Delta t} - \frac{\partial}{\partial t}\Sigma_0(x,t) \right| < \varepsilon ,$$

uniformly in x a.e. on Q_1, provided that $|\Delta t| < \delta$, where $\delta = \delta(\varepsilon, t)$, does not depend on x. Finally, we obtain from assumption a.1) and from Lemma 4.2

$$\left\| \frac{d}{dt} B_0(t) f_1 \right\|_1 \leqslant v(\gamma^* + \Sigma_0^*) \|f_1\|_1 . \tag{4.2}$$

ii) Can be proved in an analogous way and, in particular, we have

$$\|B_1(t) f_1\|_2 \leqslant \bar{k}_0 \bar{b} \|f_1\|_1 , \qquad \left\| \frac{d}{dt} B_1(t) f_1 \right\|_2 \leqslant (k_0^* \bar{b} + \bar{k}_0 b^*) \|f_1\|_1 , \tag{4.3}$$

where $\bar{b} = \max b(t)$, $b^* = \max |\dot{b}(t)|$, $t \in [0, t']$.

iii) Follows from i) and ii) and we have

$$\|B(t) f\| \leqslant [v(\bar{\gamma}_0 + \bar{\Sigma}_0) - h\bar{b}] \cdot \|f\| . \tag{4.4}$$

Lemma 4.4. *The nonlinear operator* $F(t, f)$ *is a* c.m. *on* $[0, t'] \times X$ *to* X *and locally Lipschitz continuous in* f.

Proof. By putting

$$F_1(f) = \begin{bmatrix} -f_2 f_1 \\ 0 \end{bmatrix}, \quad F_2(f) = \begin{bmatrix} f_2 J f_1 \\ 0 \end{bmatrix}, \quad F_3(f) = \begin{bmatrix} 0 \\ f_2 B_1 f_1 \end{bmatrix},$$

we then have

$$F(t, u) = v\alpha_1 \Sigma_0(x, t) F_1(f) + v\alpha_2 \gamma_0(x, t) F_2(f) + b(t) k_0(x, t) F_3(f) \alpha_3 .$$

By taking in account that

$$\|F_i(f) - F_i(g)\| \leqslant \tfrac{1}{2} C[\|f_2 - g_2\|_2(\|f_1\|_1 - \|g_1\|_1) + \|f_1 - g_1\|_1(\|f_2\|_2 + \|g_2\|_2)] ,$$

where $C = \bar{k}_0$ if $i = 1, 2$, and $C = h$ if $i = 3$. Thus, by using Lemma 4.2, Lemma 4.3 and assumption a.1), we obtain after some manipulations

$$\|F(t, f) - F(t, g)\| \leqslant \alpha(\|f\| + \|g\|)\|f - g\| \tag{4.5}$$

and, since $F(t, 0) = 0$,

$$\|F(t, f)\| \leqslant \alpha \|f\|^2 , \tag{4.6}$$

where

$$\alpha = \tfrac{1}{2}\left(\frac{v(|\alpha_1|\bar{\Sigma}_0 + |\alpha_2|\bar{\gamma}_0)}{h} + \bar{b}|\alpha_3|\right) .$$

Given $f_0 \in X$, if we consider the sphere

$$\bar{S}(f_0, r) = \{f \in X : \|f_0 - f\| \leqslant r\} ,$$

we then obtain from (4.5) and (4.6):

$$\|F(t, f) - F(t, g)\| \leqslant 2M\|f - g\| , \qquad \|F(t, f)\| \leqslant M\|f\|^2 , \tag{4.7}$$

for every $f, g \in \bar{S}(f_0, r)$ since $\|f\|$ and $\|g\|$ are not larger than $M/\alpha = (r + \|f_0\|)$. Finally, by using assumption a.1), we obtain

$$\|F(t + \Delta t, f + \Delta f) - F(t, f)\| \leqslant$$
$$\leqslant v|\alpha_1|[\max_x |\Sigma_0(x, t + \Delta t) - \Sigma_0(x, t)| \cdot \|F_1(f + \Delta f)\| + \|F_1(f + \Delta f) - F_1(f)\|\bar{\Sigma}_0] +$$
$$+ v|\alpha_2|[\max_x |\gamma_0(x, t + \Delta t) - \gamma_0(x, t)| \cdot \|F_2(f + \Delta f)\| + \|F_2(f + \Delta f) - F_2(f)\|\bar{\gamma}_0] +$$
$$+ |\alpha_3|[\max_x |b(t + \Delta t) k_0(x, t + \Delta t) - b(t) k_0(x, t)| \cdot$$
$$\cdot \|F_3(f + \Delta f)\| + \|F_3(f + \Delta f) - F_3(f)\|\bar{b}\bar{k}_0],$$

and the continuity of $F(t, f)$ follows from assumption a.3), (4.5) and (4.6) which are valid also for $\|F_i(f) - F_i(g)\|$.

Further properties of $F(t, f)$ are stated in the following

Lemma 4.5. *$F(t, f)$ has continuous derivatives F_t and F_f where F_f means the Fréchet derivative at any $f \in X$* [**11**].

Proof. In fact, after a few manipulations we obtain

$$F(t, f + g) - F(t, f) = F_f(t)\, g + w(t, g)\,,$$

where

$$F_f(t)g = \begin{bmatrix} -v\, \Sigma_0(x, t)\, \alpha_1 f_2 I + v\gamma_0(x, t)\alpha_2 f_2 J & -v\Sigma_0(x, t)\, \alpha_1 f_1 I + J f_1 \\ \alpha_3 f_1\, B_1(t) & \alpha_3 B_1(t) f_1 \end{bmatrix} \begin{bmatrix} g_1 \\ g_2 \end{bmatrix}$$

is a linear operator and where

$$w(t, g) = \begin{bmatrix} -v\alpha_1\, \Sigma_0(x, t) g_1 g_2 + v\alpha_2 \gamma_0(x, t)\, g_2 J g_1 \\ g_2 B_1(t)\, g_1 \end{bmatrix}$$

is such that $\|w(t, g)\| \leqslant 2\alpha \|g\|^2$. Our assertion then follows from assumptions a.4) and Lemma 4.3.

Combining, Lemma 4.4 and Lemma 4.5 with iii) of Lemma 4.3 we obtain

Lemma 4.6. *$G(t, f) = B(t)f + F(t, f)$ is a c.m. on $[0, t'] \times X$ to X and locally Lipschitz continuous in f, has continuous derivatives G_t and G_f. We also have*

$$G_t = \frac{\mathrm{d}}{\mathrm{d}t} B(t) f + F_t\,, \qquad G_f = B(t) + F_f\,, \tag{4.8}$$

$$\|G(t, f) - G(t, g)\| [v(\bar{\gamma}_0 + \bar{\Sigma}_0) + h\bar{b} + 2M] \|f - g\| \qquad f, g \in S(f_0, r)\,. \tag{4.9}$$

5. - An upper bound for t' and $\|u(t)\|$.

The goal of the last part of this paper is to evaluate an upper bound for t' and to find an upper bound for the continuous and non-negative function $\|u(t)\|$. To this purpose we define the following operator

$$\left\{ \begin{array}{l} P(w) = w_0(t) = \int_0^t \exp\left[(t - s)A\right] \left\{ B(s)\, w(s) + F(s, w(s)) \right\} \mathrm{d}s \\ D(P) = S(w_0, r) = \left\{ w \in Y \colon \|\!|w_0 - w|\!\| \leqslant r \right\} \subset Y\,. \end{array} \right. \tag{5.1}$$

Theorem 5.1. *If we choose* $t' < \beta_1^{-1}$ *where* $\beta_1 = \beta(r + |||w_0|||)/r$ *and* $\beta = [v(\bar{\gamma}_0 + \bar{\Sigma}_0) + h\bar{b} + 2\alpha(r + |||w_0|||)]$, *then the equation* $P(u) = u$ *has a unique solution* $u \in S(w_0, r)$ *such that* $|||u_n - u||| \to 0$ *as* $n \to \infty$, *where* $u_0 = w_0$, $u_{n+1} = P(u_n)$, $n = 0, 1, 2, \ldots$.

Proof. Let $w_i \in S(w_0, r)$, $i = 1, 2$. We have $|||w_i||| \leq (r + |||w_0|||)$ and consequently

$$\sup\left\{\|P(w_1)(t) - P(w_2)(t)\|, t \in [0, t']\right\} = |||P(w_1) - P(w_2)||| \leq t'\beta\,|||w_1 - w_2|||\,, \tag{5.2}$$

where we used Lemma 4.3 and Lemma 4.4. In an analogous way, we have

$$\begin{aligned} |||P(w_1) - w_0||| &\leq t'[v(\bar{\gamma}_0 + \bar{\Sigma}_0) + h\bar{b} + \alpha(r + \|w_0\|)]\,|||w_1||| \\ &< t'\beta\,\frac{(r + \|w_0\|)}{r}\,r \qquad\qquad \forall w_1 \in S(w_0, r). \end{aligned} \tag{5.2$'$}$$

Therefore, if we choose t' such that $t'\beta_1 < 1$, (hence, also $t'\beta < 1$), the nonlinear operator (5.1) is a contraction from $S(w_0, r)$ into itself and our assertion follows from the contraction mapping theorem. We shall now find an upper bound for $\|u(t)\|$. From (3.2) and Lemmas 4.1-4.4, we obtain the following inequality

$$\|u(t)\| \leq \|w_0(t)\| + \int_0^t \left\{z\|u(s)\| + \alpha\|u(s)\|^2\right\} \mathrm{d}s\,,$$

where $z = [v(\bar{\gamma}_0 + \bar{\Sigma}_0) + h\bar{b}]$, $\|w_0(t)\| \leq \|u_0\| + \|v_0\|\,t$, $t \in [0, t']$.

Inequality suggests that we investigate whether or not the integral equation

$$y(t) = \|u_0\| + t\|v_0\| + \int_0^t \left\{zy(s) + \alpha[y(s)]^2\right\} \mathrm{d}s \tag{5.3}$$

has a continuous non-negative solution. To this end, we differentiate equation (5.3) and we have

$$\frac{\mathrm{d}y}{\mathrm{d}t} = [y(t)]^2 + zy(t) + \|v_0\|\,, \qquad y(0) = \|u_0\|\,. \tag{5.4}$$

Just as in [**3**]$_3$, we have that the solution of (5.4)

$$y(t) = \frac{\Delta - z}{2\alpha} + \frac{1}{\varphi(t)},$$

$$\varphi(t) = -\frac{\alpha}{\Delta}\left[\frac{\Delta + (z + 2\alpha\|u_0\|)}{\Delta - (z + 2\alpha\|u_0\|)}\exp[-\Delta t] + 1\right], \qquad \Delta = (z - 4\alpha\|v_0\|)^{\frac{1}{2}}$$

is a continuous non-negative and finite function at any $t \in [0, t_1]$, where t_1 is the first positive root of $\varphi(t)$, provided that α is sufficiently close to zero. Consequently, if we choose $t_0 < \min(t', t_1)$ then $u(t)$ and $y(t)$ exist and by applying the successive approximation method of Theorem 5.2 we have

$$\|u(t)\| \leqslant y(t), \qquad t \in [0, t_0].$$

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Sommario.

Facendo uso di alcuni risultati della teoria delle equazioni non lineari di evoluzione negli spazi di Banach, si studia un problema non lineare di trasporto di neutroni.

Si prova l'esistenza e l'unicità di una soluzione $u = u(t)$ *di tipo « mild » e quindi si mostra che tale soluzione è anche di tipo « strict ».*

Infine, si esaminano alcune proprietà della norma $\|u(t)\|$, *che hanno interesse per le applicazioni.*

Summary.

The object of this paper is to study a nonlinear neutron transport initial-value problem in a suitable Banach space X.

By using some results of the theory of nonlinear evolution equations of hyperbolic type in Banach spaces, we prove the existence and uniqueness of a local « mild solutions » $u = u(t)$ *belonging to* X *and we then show that such a solution is also a « strict solution ». Finally, we investigate some properties of* $\|u(t)\|$ *which are of practical interest.*

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