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The brachistochronic motion of a material point on surface. (**)

1. - Introduction.

In present time there is an increasing scientific interest in problems involving the optimization of mechanical systems during the motion, although this is in fact a very old problem. Indeed, John Bernoulli (1696) first formulated and solved a problem of this kind, the so called brachistochrone problem:

Suppose that in a vertical plane two points A and B are given. We wish to determine the shape of a smooth tube joining A and B, and such that a mass point M sliding in the tube under influence of gravity, and starting from A with a given speed, shall reach B in the shortest possible time.

The Bernoulli's brachistochrone problem have been generalized, for the holonomic and nonholonomic mechanical systems of n degrees of freedom, by Pennachietti [1], Mc Connell [2] and Djukic [7]. Also, there exist a group of solved particular brachistochrone problems as: the brachistochrone with dry friction [3], brachistochrone in a resisting medium [6] (p. 241) and [5] and brachistochrone in a central force field [4].

In this Note we will consider the brachistochrone motion of a particle on a smooth surface under the influence of gravity. The results of the papers [1], [2] and [7] will be slightly modified and used for solving the two particular problems.

In this paper the summation convention will be observed. Small italic indices imply a range of values 1, 2 and 3.

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2. - Analysis.

Let us consider the motion of a particle with coordinates q^i, whose kinetic energy is given by the equation

$$T = g_{ij}(q^r)\dot{q}^i\dot{q}^j/2 , \tag{1}$$

$\dot{q}^i$ being the derivative of q^i with respect to time t and g_{ij} being a function of the q's only. The particle is acted upon by a conservative field of force of potential energy $\pi(q^i)$. The particle is free to move on a smooth surface, whose equation is given by

$$F(q^i) = 0 . \tag{2}$$

During any motion of the particle the energy equation

$$T + \pi = h , \tag{3}$$

where h is a constant, is valid.

Suppose that on the surface (2) two points A and B are given. We wish to determine the shape of a smooth tube joining A and B, which is situated on the surface (2), and such that a particle M sliding in the tube under the influence of force of potential π, and starting from A with a given speed, shall reach B in the shortest possible time, that is, for a minimal value of the functional

$$I = \int_0^\tau \mathrm{d}t . \tag{4}$$

Solving (1) and (3) with respect to dt, and substituting this result into (4) we obtain the problem of minimizing the following functional

$$I = \int_0^\tau T_1^{1/2}\,\mathrm{d}t , \tag{5}$$

subject to the holonomic constraint (2), where

$$T_1(q^i, \dot{q}^i) = \frac{T(q^i, \dot{q}^i)}{h - \pi(q^i)} \tag{6}$$

and where $T_1 = 1$ during the motion. The variational problem is equivalent to the minimization of a new functional

$$I^* = \int_0^\tau \{[T_1(q^i, \dot{q}^i)]^{1/2} + \lambda F(q^i)\}\, dt\,, \tag{7}$$

where the λ is Lagrange's multiplier.

In brachistochronic motion the first variation of (7), where the terminal time τ is not specified, must be zero. This condition, $\delta I^* = 0$, together with $\delta q^i(0) = 0$, $\delta q^i(\tau) = 0$, $\delta t(\tau) \neq 0$ and (1) yields (for more details see [**6**] p. 222)

$$\frac{d}{dt}\frac{\partial T_1}{\partial \dot{q}^i} - \frac{\partial T_1}{\partial q^i} = 2\lambda \frac{\partial F}{\partial q^i}\,. \tag{8}$$

Hence, we have the following theorem:

Brachistochronic motion, with constant energy (3) of a material point on a smooth surface (2) is described by the differential equations (8) and the algebraic equation (2).

Six constants of integration of the system (8) and (2) and the terminal time τ of the brachistochronic motion may be found from the six equations

$$q^i(0) = q^{i0}\,, \qquad q^i(\tau) = q^{i1}\,, \tag{9}$$

and the equation $T_1 = 1$ during the motion. Here q^{i0} and q^{i1} are the coordinates of the particle at initial and terminal time. These quantities are satisfying the surface equation (2) identically.

3. - Brachistochronic motion of a particle on a sphere.

Let us consider brachistochronic motion of a material point of unit mass in a smooth tube, which is situated on a sphere's surface, under the influence of gravity. In this case the kinetic and potential energy are

$$T = \tfrac{1}{2}(\dot{\varrho}^2 + \varrho^2\dot{\theta}^2 + \dot{z}^2), \qquad \pi = -gz, \tag{10}$$

where g is the acceleration of gravity, ϱ, θ and z are cylindrical coordinates of the point, where z-axis is oriented vertically downward and where the origin is situated in the sphere's center. The equation of the surface (2) is

$$F \equiv \varrho^2 + z^2 - a^2 = 0\,, \tag{11}$$

where a is the radius of sphere.

Substituting (10), (11) and (6) into (8) we obtain the following differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{\varrho}}{h+gz}\right)-\frac{\varrho\dot{\theta}^2}{h+gz}=4\lambda\varrho\,, \tag{12}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{z}}{h+gz}\right)+\frac{g}{h+gz}=4\lambda z\,, \tag{13}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\varrho^2\dot{\theta}}{h+gz}\right)=0\,. \tag{14}$$

Integrating the equation (14) we have

$$\varrho^2\dot{\theta}=c_1(h+gz)\,,\qquad c_1=\text{const.}\,, \tag{15}$$

and combining this result with (10), (11) and (3) yields

$$a^2\dot{z}^2=f(z)\,, \tag{16}$$

where

$$f(z)=(h+gz)[2(a^2-z^2)-c_1^2(h+gz)]\,. \tag{17}$$

The polynomial $f(z)$ is negative for $z=-a$ and $+a$, positive for $z=-\infty$, and negative for $z=\infty$. From (16), it is obvious that $f(z)$ must be equal or greater from zero during the motion. Therefore the polynomial $f(z)$ has the form

$$f(z)=-g(z-\alpha)(z-\beta)(z-\gamma)\,, \tag{18}$$

where its roots are satisfying $-\infty<\gamma<-a<\beta<\alpha<a<\infty$. Hence, it is evident that, during the motion, z is between the values β and α, i.e.

$$\beta\leqslant z\leqslant\alpha\,. \tag{19}$$

Combining now (16), (18) and (19) we have the result

$$\frac{\sqrt{g(\alpha-\gamma)}}{2a}\,t+c_2=F(\varphi,k)\,, \tag{20}$$

where

$$k^2\equiv\frac{\alpha-\beta}{\alpha-\gamma}>0\,,\qquad z=\alpha-(\alpha-\beta)\sin^2\varphi\,,\qquad c_2=\text{const.}\,, \tag{21}$$

and where $F(\varphi, k)$ is [8] the elliptic integral of the first order. Substituting (16), (18), (19) and (21) into (15) we obtain complet solution of the problem

$$\frac{1}{c_1}\sqrt{\frac{\alpha-\gamma}{g}}\,\theta + c_3 = \frac{h/g - a}{a+\alpha}\,\pi\left(\varphi, \frac{\beta-\alpha}{a+\alpha}, k\right) + \tag{22}$$

$$+ \frac{h/g + a}{a-\alpha}\,\pi\left(\varphi, \frac{\alpha-\beta}{a-\alpha}, k\right), \qquad c_3 = \text{const.},$$

where $\pi(\varphi, n, k)$ is [8] the elliptic integral of the third order.

4. - Brachistochronic motion of a particle on the paraboloid.

Let us consider the brachistochronic motion of a particle of unit mass on the paraboloid, whose equation is

$$F \equiv \varrho^2 - 2az = 0\,, \tag{23}$$

where a is a constant, under the influence of gravity.

In this case the kinetic and potential energy of the particle are

$$T = \tfrac{1}{2}(\dot\varrho^2 + \varrho^2\dot\theta^2 + \dot z^2)\,, \qquad \pi = gz\,, \tag{24}$$

where ϱ, θ and z are the cylindrical coordinates of the point. Hence, the differential equation (8) of the brachistochronic motion are

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\varrho^2\dot\theta}{h-gz}\right) = 0\,; \qquad \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\dot\varrho}{h-gz}\right) - \dfrac{\varrho\dot\theta^2}{h-gz} = 4\varrho\lambda \\[2ex] \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\dot z}{h-gz}\right) - \dfrac{g}{h-gz} = -\,4\lambda a\,. \end{cases} \tag{25}$$

From the first equation we have immediately

$$\varrho^2\dot\theta = c_1(h-gz)\,, \qquad c_1 = \text{const.} \tag{26}$$

and combining this result with (23), (24) and (3) the following equation

$$\left(z + \frac{a}{2}\right)\dot z^2 = (h-gz)\left[2z - \frac{c_1^2}{2}(h-gz)\right]. \tag{27}$$

Solution to this equation is given by

$$vt = c_2 - kE\left(\alpha, \frac{\sqrt{k^2-1}}{k}\right), \qquad c_2 = \text{const.}, \tag{28}$$

where $E(\alpha, n)$ is [**8**] the elliptic integral of the second order, and

$$\alpha = \text{arc sin}\,\frac{k\cos\varphi}{(k^2-1)^{\frac{1}{2}}}, \qquad z = \left(\frac{h}{g}+\frac{a}{2}\right)\sin^2\varphi - \frac{a}{2}, \tag{29}$$

$$k^2 = 1 + \frac{2h}{g[a + c_1^2/2\,(h + g\,a/2)]}, \quad \nu = \frac{2ghk^2}{(2h+ag)\{[2h-ag(k^2-1)](k^2-1)\}^{\frac{1}{2}}}, \tag{30}$$

where $k^2 > 1$ for $h > 0$.

Using (26), (30) we obtain solution for the coordinate θ in the form

$$\mu k\theta + c_3 = -\frac{2h}{2h+ag}\left[F\left(\alpha, \frac{\sqrt{k^2-1}}{k}\right) + \right. \tag{31}$$

$$\left. + \frac{ag}{2h}\pi\left(\alpha, \frac{2h+ag}{2hk^2}(k^2-1), \frac{\sqrt{k^2-1}}{k}\right)\right] + E\left(\alpha, \frac{\sqrt{k^2-1}}{k}\right),$$

where $c_3 = \text{const.}$ and

$$\mu = \frac{2ah\sqrt{g}}{[2h-ag(k^2-1)]\sqrt{2h+ag}}. \tag{32}$$

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Abstract.

The solutions for the brachistochronic motion of a material point under the influence of gravity on the smooth sphere and paraboloid are presented.

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