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On common fixed points through rational expression. (**) 

1. - Introduction.

Let \((X, d)\) be a complete metric space. A mapping \(f: X \to X\) is called a contraction if there exists a real number \(k\), \(0 < k < 1\), such that

\[ d(f(x), f(y)) \leq kd(x, y), \quad x, y \in X. \]

The celebrated Banach's contraction principle [1] states that every such mapping admits a unique fixed point. The condition that the mapping is a contraction is a very severe restriction as also such maps are uniformly continuous. Kannan [2] established the same theory for a mapping \(f\) which satisfies

\[ d(f(x), f(y)) \leq \alpha [d(x, f(x)) + d(y, f(y))], \quad 0 < \alpha < \frac{1}{3}, \]

and showed that \(f\) need not be even continuous. Later on Reich [4] and Wong [5] established the same results through generalized contractions.

In this paper we have made an attempt to establish a fixed point theorem through rational expression for product of two self-mappings defined on a metric space (not necessarily complete). Also the mappings under study need not be continuous. Some results which follow as its consequences are also derived.

We have

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Theorem 1. Let $T_1$ and $T_2$ be two self-mappings defined on a metric space $(X, d)$ such that

(i) $d(T_1 T_2(x), T_2 T_1(y)) \leq \frac{\beta d(x, T_1 T_2(x)) [\alpha + d(y, T_2 T_1(y))]}{\alpha + \beta d(x, T_1 T_2(x))}$,

for all $x, y \in X$, $\alpha > 0$, $0 < \beta < 1$,

(ii) for some $x \in X$, the sequence $\{x_n\}$ defined as $x_1 = T_1 T_2(x)$ and for $n > 1$,

\[
x_n = \begin{cases} T_1 T_2(x_{n-1}) & \text{if } n \text{ is odd} \\ T_2 T_1(x_{n-1}) & \text{if } n \text{ is even} \end{cases}
\]

has a subsequence $\{x_{n_k}\}$ with $x_0 = \lim_{n \to \infty} x_n$. Then $x_0$ is the unique common fixed point of $T_1$ and $T_2$.

Proof. For the sequence $\{x_n\}$ as defined in (ii) we have

\[
d(x_1, x_2) = d(T_1 T_2(x), T_2 T_1(x)) \leq \frac{\beta d(x, T_1 T_2(x)) [\alpha + d(x_1, T_2 T_1(x_1))]}{\alpha + \beta d(x, T_1 T_2(x))},
\]

i.e.

\[
\left[ 1 - \beta d(x, x_1) \right] d(x_1, x_2) \leq \frac{\alpha \beta d(x, x_1)}{\alpha + \beta d(x, x_1)},
\]

which gives $d(x_1, x_2) < \beta d(x, x_1)$.

Also

\[
d(x_2, x_3) = d(T_2 T_1(x_1), T_1 T_2(x_2)) \leq \frac{\beta d(x_2, T_1 T_2(x_2)) [\alpha + d(x_3, T_2 T_1(x_3))]}{\alpha + \beta d(x_2, T_1 T_2(x_2))},
\]

i.e., $\alpha + \beta d(x_2, x_3) < \alpha \beta + \beta d(x_1, x_2)$ which implies

\[
d(x_2, x_3) < \frac{\alpha}{\beta} (\beta - 1) + \beta d(x, x_1).
\]
Again
\[ d(x_n, x_{n+1}) = d(T(x_n), T(x_{n+1})) \]
\[ = \frac{\alpha d(x_n, T(x_n)) + \beta d(x_n, T(x_{n+1}))}{\alpha + \beta d(x_n, T(x_n))} \]
which implies
\[ d(x_n, x_{n+1}) \leq \beta d(x_n, x_{n+1}) < \alpha (\beta - 1) + \beta^2 d(x_n, x_{n+1}), \]
and
\[ d(x_n, x_{n+1}) = d(T(x_n), T(x_{n+1})) \]
\[ = \frac{\alpha d(x_n, T(x_n)) + \beta d(x_n, T(x_{n+1}))}{\alpha + \beta d(x_n, T(x_n))} \]
i.e., \( \alpha + \beta d(x_n, x_n) < \alpha \beta + \beta d(x_n, x_n) \) which implies
\[ d(x_n, x_{n+1}) \leq \frac{\alpha}{\beta} (\beta - 1) + \alpha (\beta - 1) + \beta^2 d(x_n, x_{n+1}) = \frac{\alpha}{\beta} (\beta^2 - 1) + \beta^3 d(x_n, x_{n+1}). \]

In general,
\[ d(x_{2n}, x_{2n+1}) \leq \frac{\alpha}{\beta} (\beta^n - 1) + \beta^n d(x_n, x_{n+1}) \]
and
\[ d(x_{3n+1}, x_{3n+2}) < \alpha (\beta^n - 1) + \beta^{n+1} d(x_n, x_{n+1}). \]

Hence for \( m = 2r \), we have
\[ d(x_m, x_{m+n}) < d(x_m, x_{m+1}) + d(x_{m+1}, x_{m+2}) + \ldots + d(x_{m+n-1}, x_{m+n}) \leq \]
\[ \leq \frac{\alpha}{\beta} (\beta^r - 1) + \beta^r d(x_n, x_{n+1}) + \alpha (\beta^r - 1) + \beta^{r+1} d(x_n, x_{n+1}) + \ldots \quad (2n \text{ terms}) \]
\[ = \frac{\alpha}{\beta} (\beta^r - 1) + \alpha (\beta^r - 1) + \frac{\alpha}{\beta} (\beta^{r+1} - 1) + \alpha (\beta^{r+1} - 1) + \ldots \quad (n \text{ terms}) + \]
\[ + (\beta^r + 2\beta^{r+1} + 2\beta^{r+2} + \ldots, \quad (n \text{ terms}) d(x, x_n) \]
\[ < \alpha \beta^{r-1}(1 + \beta + \beta^2 + \ldots) + \alpha \beta^r(1 + \beta + \beta^2 + \ldots) + \]
\[ + 2\beta^r(1 + \beta + \beta^2 + \ldots) d(x, x_n) \]
\[ = \frac{\alpha \beta^{r-1}(1 + \beta)}{1 - \beta} + \frac{2\beta^r}{1 - \beta} d(x, x_n) \rightarrow 0 \text{ as } r \rightarrow \infty, \quad \text{i.e., } m \rightarrow \infty. \]
Similarly, for $m = 2r + 1$, we have

$$d(x_m, x_{m+1}) \to 0 \quad \text{as} \quad m \to \infty,$$

showing thereby that $\{x_n\}$ is a Cauchy sequence. As a consequence of (ii), there exists $x_0 \in X$ such that

$$\lim_{n \to \infty} x_n = x_0.$$

We shall show that $T_1T_2(x_0) = T_2T_1(x_0) = x_0$. Now

$$d(T_1T_2(x_0), x_0) < d(T_1T_2(x_0), x_t) + d(x_t, x_0) \quad \text{(where t is even)}$$

$$= d(T_1T_2(x_0), T_2T_1(x_{t-1})) + d(x_t, x_0)$$

$$\leq \frac{\beta d(x_0, T_2(x_0)[x + d(x_{t-1}, T_2T_1(x_{t-1})] + d(x_t, x_0)}{\alpha + \beta d(x_0, T_1T_2(x_0))},$$

i.e.,

$$d(T_1T_2(x_0), x_0) < d(x_t, x_0) \left(1 - \frac{\alpha \beta + \beta d(x_{t-1}, x_t)}{\alpha + \beta d(x_0, T_1T_2(x_0))} \right)^{-1} \to 0$$

for sufficiently large $t$. Thus $T_1T_2(x_0) = x_0$. Similarly $T_2T_1(x_0) = x_0$.

Next, we show that $x_0$ is the unique fixed point of $T_1T_2$ and $T_2T_1$. Let $y_0$ also satisfy $T_1T_2(y_0) = T_2T_1(y_0) = y_0$. Then

$$d(x_0, y_0) = d(T_1T_2(x_0), T_2T_1(y_0)) < 0.$$

Therefore $x_0 = y_0$. It can be easily seen that $x_0$ is the only fixed point of $T_1T_2$ and $T_2T_1$ which is common. Next, we show that $T_1(x_0) = x_0 = T_2(x_0)$.

We have $T_1T_2(x_0) = x_0 = T_2T_1(x_0)$ which implies $T_1(T_2T_1(x_0)) = T_1(x_0)$ i.e., $T_1T_2T_1(x_0) = T_1(x_0)$. Therefore $T_1(x_0) = x_0$, since $x_0$ is the unique fixed point of $T_1T_2$. Similarly $T_2(x_0) = x_0$. Lastly, we show that $x_0$ is the only common fixed point of $T_1$ and $T_2$. Let $z_0 \neq x_0$ be an element of $X$ enjoying the property $T_1(z_0) = T_2(z_0) = z_0$ which implies $T_1T_2(z_0) = T_1T_1(z_0) = z_0$ i.e., $z_0 = x_0$, since $x_0$ is the unique common fixed point of $T_1T_2$ and $T_2T_1$. This completes the proof of the theorem.

Remark. The result by setting $T_2x = x$ for all $x \in X$ in the above theorem, has been proved independently in [3].
2. – In the following theorem we show that if \( T_1 \) and \( T_2 \) are continuous, then it is sufficient for the validity of Theorem 1 that \( T_1, T_2 \) satisfy condition (i) only on a dense subset of \( X \).

**Theorem 2.** Let \( T_1 \) and \( T_2 \) be two continuous mappings of \( X \) into itself such that for any pair of points \( x, y \) belonging to an everywhere dense subset \( M \) of \( X \)

\[
(\ast) \quad d(T_1 T_2(x), T_2 T_1(y)) < \frac{\beta d(x, T_1 T_2(x))[\alpha + d(y, T_2 T_1(y))]}{\alpha + \beta d(x, T_1 T_2(x))}
\]

\( \alpha > 0, \ 0 < \beta < 1, \) and (ii) holds, then \( T_1 \) and \( T_2 \) have a unique common fixed point.

**Proof.** The proof will follow from Theorem 1 if we show that the expression in (\( \ast \)) holds for any pair of points \( x, y \in X \).

Let \( x, y \in X \). If \( x \in X \), \( y \in X^\sim M \), then a sequence \( \{y_n\} \) in \( M \) converges to \( y \) and we have

\[
d(T_1 T_2(x), T_2 T_1(y)) < d(T_1 T_2(x), T_2 T_1(y_n)) + d(T_2 T_1(y_n), T_2 T_1(y)) \\
< \frac{\beta d(x, T_1 T_2(x))[\alpha + d(y_n, T_2 T_1(y_n))]}{\alpha + \beta d(x, T_1 T_2(x))} + d(T_2 T_1(y_n), T_2 T_1(y)) \\
= \frac{\beta d(x, T_1 T_2(x))[\alpha + d(y, T_2 T_1(y))]}{\alpha + \beta d(x, T_1 T_2(x))},
\]

since \( T_1 \) and \( T_2 \) are continuous.

Now we consider the case when \( x, y \in X^\sim M \). Then there exist sequences \( \{x_n\} \) and \( \{y_n\} \) in \( M \) converging to \( x \) and \( y \) respectively. Again we shall have condition (i) using the continuity of \( T_1 \) and \( T_2 \) and the inequality

\[
d(T_1 T_2(x), T_2 T_1(y)) < d(T_1 T_2(x), T_1 T_2(x_n)) + \\
+ d(T_1 T_2(x_n), T_2 T_1(y_n)) + d(T_2 T_1(y_n), T_2 T_1(y)) .
\]

3. – A much more interesting situation when a function which is the limit of a convergent sequence of functions is shown to have unique fixed point which is the limit of the fixed points of the sequence of functions is handled in the next theorem.

**Theorem 3.** Let \( \{T_1, T_2\} \) and \( \{T_2, T_1\} \) be two sequences of mappings of \( X \) into itself converging pointwise to \( T_1 T_2 \) and \( T_2 T_1 \) respectively such that

\[
d((T_1 T_2)(x), (T_2 T_1)(y)) < \frac{\beta d(x, (T_1 T_2)(x))[\alpha + d(y, (T_2 T_1)(y))]}{\alpha + \beta d(x, (T_1 T_2)(x))}
\]

for all \( x, y \in X, \ \alpha > 0, \ 0 < \beta < 1, \ i = 1, 2, \ldots. \)
If \((T_1, T_2)_n\) and \((T_2, T_1)_n\) have common fixed point \(\xi_1\) and \(\xi\) is the common fixed point of \(T_1, T_2\) and \(T_2, T_1\), then the sequence \(\xi_n\) converges to \(\xi\).

**Proof.** We have

\[
\begin{align*}
d(\xi, \xi_n) &= d(T_1, T_2)(\xi, (T_2, T_1)_n(\xi)) \\
&< d((T_1, T_2)_n(\xi), (T_2, T_1)_n(\xi)) + d((T_2, T_1)_n(\xi), (T_1, T_2)_n(\xi)) \\
&< d(T_1, T_2)(\xi, (T_1, T_2)_n(\xi)) + \frac{\beta d(\xi, (T_1, T_2)_n(\xi)) + d(\xi_n, (T_1, T_2)_n(\xi))}{\alpha + \beta d(\xi, (T_1, T_2)_n(\xi))} \\
&= \left\{ 1 + \frac{\alpha \beta}{\alpha + \beta d(T_1, T_2)(\xi, (T_1, T_2)_n(\xi))} \right\} d(T_1, T_2)(\xi, (T_1, T_2)_n(\xi)) \rightarrow 0
\end{align*}
\]

as \((T_1, T_2)_n(\xi) \rightarrow T_1, T_2(\xi)\). Hence \(\xi_n\) converges to \(\xi\).

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**References.**


**Abstract.**

The well known Banach’s contraction mapping theorem has been generalized by various authors through generalized contractions. In this paper, an attempt has been made to derive certain results pertaining to fixed point theorems in metric spaces through a rational expression.

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