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On extreme rates of growth of functions analytic in a finite disc. (**)

1. – Let U_R $(0 < R \leqslant \infty)$ denote the class of functions analytic in the disc $D_R \equiv \{z\colon |z| < R\}$, which are not polynomials or rational functions. The maximum modulus $M(r, f)$ of a function $f(z) \in U_R$ is as usual defined as $M(r, f) = \max_{|z|=r} |f(z)|$ $(0 < r < R)$. It is well known (see e.g. [1]$_1$, [2] p. 73, [3], [4]) that there exist functions in the class U_R whose rates of growth, as measured by their maximum moduli are arbitrarily fast or arbitrarily slow. For $R = \infty$, Lepson [1]$_1$ proved the following by means of simple constructions:

Theorem A. Let $h(r)$ and $k(r)$ be positive functions of r for $r > 0$ such that $\log k(r) \neq 0(\log r)$ as $r \to \infty$. Then there exists a function $f(z)$ with nonnegative coefficients, belonging to U_∞ and two sequences $\{c_n\}$ and $\{r_n\}$ of positive numbers tending monotonically to ∞ such that for every positive integer n, $M(c_n, f) > h(c_n)$ and $M(r_n, f) < k(r_n)$.

Theorem B. Let $h(r)$ be a positive function of r for $r > 0$ bounded in every finite interval. Then there exists a function $f(z)$ with nonnegative coefficients, belonging to U_∞, such that $M(r, f) > h(r)$ for all r.

Theorem C. Let $k(r)$ be a positive function of r having a positive lower bound for $r > 0$ and such that $(\log k(r)/\log r) \to \infty$ as $r \to \infty$. Then there exists a function $f(z)$ with nonnegative coefficients, belonging to U_∞, such that $M(r, f) < k(r)$ for all r.

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In the present paper, using the above results, we prove the existence of similar functions in the class U_R $(0<R<\infty)$. Theorem 1 shows the existence of a function belonging to U_R $(0<R<\infty)$ whose upper rate of growth is arbitrarily fast and simultaneously whose lower rate of growth is arbitrarily slow. Further, in Theorem 4, we show that there exists a function in the class U_R $(0<R<\infty)$ which has two arbitrarily prescribed rates of growth on two different, although unspecified sequences tending to R. As corollaries to Theorems 1 to 4 we also show the existence of similar functions belonging to the class U^c_R consisting of those functions which are analytic in $D^c_R \equiv \{z\colon |z|>R\}$ $(0<R<\infty)$.

2. – Theorem 1. *Let $\lambda(r)$ and $\mu(r)$ be positive functions of r for $0\leqslant r<R<\infty$ such that $\log\lambda(r)\neq 0(-\log(R-r))$ as $r\to R^-$. Then there exists a function $F(z)$ belonging to the class U_R $(0<R<\infty)$, with nonnegative coefficients and two sequences $\{s_n\}$ and $\{t_n\}$ of positive numbers tending monotonically to R such that, for every positive integer n, $M(s_n, F)>\mu(s_n)$ and $M(t_n, F)<\lambda(t_n)$.*

Proof. Define,

$$k(s)=\begin{cases}\lambda(0) & \text{for}\quad 0<s\leqslant \dfrac{1}{R}\\[2ex] \lambda\left(R-\dfrac{1}{s}\right) & \text{for}\quad \dfrac{1}{R}<s<\infty\,,\end{cases}$$

and

$$h(s)=\begin{cases}\mu(0) & \text{for}\quad 0<s\leqslant \dfrac{1}{R}\\[2ex] \mu\left(R-\dfrac{1}{s}\right) & \text{for}\quad \dfrac{1}{R}<s<\infty.\end{cases}$$

Then $h(s)$ and $k(s)$ are positive functions of s for $0<s<\infty$ and $\log k(s)=$ $=\log\lambda(R-1/s)\neq 0(\log s)$ as $s\to\infty$. Therefore, by Theorem A, there exists a function $f(z)$ of the class U_∞ with nonnegative coefficients and two sequences $\{c_n\}$ and $\{r_n\}$ of positive numbers tending monotonically to infinity such that for every $n>0$

$$f(r_n)=M(r_n, f)<k(r_n)\,,\qquad f(c_n)=M(c_n, f)>h(c_n)\,.$$

Let N and N' be positive integers such that $r_{N-1}<1/R\leqslant r_N$ and $c_{N'-1}<1/R\leqslant c_{N'}$ and set

$$t_n=R-\frac{1}{r_{n+N}}\,,\quad s_n=R-\frac{1}{c_{n+N'}}\qquad (n=1,2,\ldots)\,.$$

Then $\{t_n\}$ and $\{s_n\}$ are sequences of positive numbers tending monotonically to R. Define,

$$F(z) = f\left(\frac{1}{R-z}\right) \quad \text{for } z \in D_R \qquad (0 < R < \infty).$$

Then $F(z) \in U_R$ $(0 < R < \infty)$ and has nonnegative coefficients. Further

$$F(t_n) = M(t_n, F) = M\left(\frac{1}{R-t_n}, f\right) = M(r_{n+N}, f) < k(r_{n+N}) = k\left(\frac{1}{R-t_n}\right) = \lambda(t_n)$$

and

$$F(s_n) = M(s_n, F) = M\left(\frac{1}{R-s_n}, f\right) = M(c_{n+N'}, f) > h(c_{n+N'}) = h\left(\frac{1}{R-s_n}\right) = \mu(s_n).$$

Hence the theorem.

Corollary. *Let $\lambda_1(r)$ and $\mu_1(r)$ be positive functions of r for $0 < R < r < \infty$ such that $\log \lambda_1(r) \neq 0(-\log(r-R))$ as $r \to R^+$. Then there exists a function $G(z)$ belonging to class U_R^c $(0 < R < \infty)$, with nonnegative coefficients, and two sequences $\{s_n'\}$ and $\{t_n'\}$ of positive numbers decreasing monotonically to R such that for every positive integer n, $M(s_n', G) > \mu_1(s_n')$ and $M(t_n', G) < \lambda_1(t_n')$.*

Proof. Let $S = 1/R$ and $u = 1/r$. Define $\lambda(u) = \lambda_1(1/u)$ and $\mu(u) = \mu_1(1/u)$. Then $\log \lambda(u) \neq 0$ $(-\log(S-u))$ as $u \to S^-$. Hence, by Theorem 1, there exist two sequences $\{s_n\}$ and $\{t_n\}$ tending to S and a function $F(z) \in U_S$ such that

$$M(t_n, F) < \lambda(t_n) \quad \text{and} \quad M(s_n, F) > \mu(s_n).$$

Put $s_n' = 1/s_n$, $t_n' = 1/t_n$ and $G(z) = F(1/z)$. Then

$$M(s_n', G) = M\left(\frac{1}{s_n'}, F\right) = M(s_n, F) > \mu(s_n) = \mu\left(\frac{1}{s_n'}\right) = \mu_1(s_n')$$

and

$$M(t_n', G) = M\left(\frac{1}{t_n'}, F\right) = M(t_n, F) < \lambda(t_n) = \lambda\left(\frac{1}{t_n'}\right) = \lambda_1(t_n').$$

By using Theorems B and C respectively the next two theorems and their corollaries follow on the lines similar to those of Theorem 1 and its corollary, hence we state them without proof.

Theorem 2. *Let $\mu(r)$ be a positive function of r for $0 \leqslant r < R < \infty$ such that it is bounded in every subinterval of $(0, R)$. Then there exists a function $F(z)$, belonging to the class U_R $(0 < R < \infty)$ with nonnegative coefficients, such that $M(r, F) > \mu(r)$ for $0 < r < R < \infty$.*

Corollary. *Let $\mu_1(r)$ be positive function of r for $0 < R < r < \infty$ such that it is bounded on every finite subinterval of (R, ∞). Then there exists a function $G(z)$ belonging to the class U_R^c $(0 < R < \infty)$, with nonnegative coefficients, such that $M(r, F) > \mu_1(r)$ for $0 < R < r < \infty$.*

Theorem 3. *Let $\lambda(r)$ be a positive function of r for $0 \leqslant r < R < \infty$ such that*

$$\lim_{r \to R^-} \frac{\log \lambda(r)}{-\log (R-r)} = \infty .$$

Then there exists a function $F(z)$ belonging to U_R $(0 < R < \infty)$, with nonnegative coefficients, such that $M(r, F) < \lambda(r)$ for $0 < r < R < \infty$.

Corollary. *Let $\lambda_1(r)$ be a positive function of r for $0 < R < r < \infty$ such that*

$$\lim_{r \to R^+} \frac{\log \lambda_1(r)}{-\log (r-R)} = \infty .$$

Then there exists a function $G(z)$ belonging to the class U_R^c $(0 < R < \infty)$, with nonnegative coefficients such that $M(r, G) < \lambda_1(r)$ for $0 < R < r < \infty$.

Theorem 4. *Let $\lambda(r)$ and $\mu(r)$ be positive and continuous functions of r for $0 < r < R < \infty$ such that $\log \lambda(r) \neq 0\big(-\log (R-r)\big)$ and $\log \mu(r) \neq 0\big(-\log \cdot (R-r)\big)$ as $r \to R^-$. Then there exists a function $F(z)$ belonging to the class U_R $(0 < R < \infty)$, with nonnegative coefficients, such that $M(r, F) = \lambda(r)$ on some sequence of values of $r \to R^-$ and $M(r, F) = \mu(r)$ on another such sequence.*

Proof. Let $\alpha(r) = \min\big(\mu(r), \lambda(r)\big)$ and $\beta(r) = \max\big(\mu(r), \lambda(r)\big)$. First let $\log \alpha(r) \neq 0\big(-\log (R-r)\big)$ as $r \to R^-$. Then by Theorem 1, there exists a function $F(z) \in U_R$ $(0 < R < \infty)$ with nonnegative coefficients and two sequences $\{s_n\}$ and $\{t_n\}$ of positive numbers tending monotonically to R such that, for every positive integer n, $M(s_n, F) > \beta(s_n)$ and $M(t_n, F) < \alpha(t_n)$. Hence $M(s_n, F) > \mu(s_n)$, $M(t_n, F) < \mu(t_n)$ and $M(s_n, F) > \lambda(s_n)$, $M(t_n, F) < \lambda(t_n)$. Thus by the continuity of $M(r, F)$, $\lambda(r)$ and $\mu(r)$, it follows that $M(r, F) = \mu(r)$ on one sequence of values tending to R and $M(r, F) = \lambda(r)$ on another such sequence.

Next suppose that $\log \alpha(r) = 0\left(-\log(R-r)\right)$ as $r \to R^-$. We can find a pair $\mu^*(r)$ and $\lambda^*(r)$ of continuous functions on $0 < r < R < \infty$, which are bounded away from zero,

$$\lim_{r\to R^-} \frac{\log \mu^*(r)}{-\log(R-r)} = \lim_{r\to R^-} \frac{\log \lambda^*(r)}{-\log(R-r)} = \infty$$

and there exist two sequences $\{s_n^*\}$ and $\{t_n^*\}$ such that

$$\mu(s_n^*) = \mu^*(s_n^*) \quad \text{and} \quad \lambda(t_n^*) = \lambda^*(t_n^*) .$$

Let $\delta(r) = \min\left(\mu^*(r), \lambda^*(r)\right)$. Then by Theorem 3, there exists a function $F(z) \in U_R$ $(0 < R < \infty)$, with nonnegative coefficients, such that $M(r, F) < \delta(r)$ for $0 < r < R < \infty$. Now

$$\lim_{n\to\infty} \frac{\log \mu(s_n^*)}{-\log(R-s_n^*)} = \lim_{n\to\infty} \frac{\log \lambda(t_n^*)}{-\log(R-t_n^*)} = \infty .$$

But

$$\frac{\log \alpha(s_n^*)}{-\log(R-s_n^*)} \quad \text{and} \quad \frac{\log \alpha(t_n^*)}{-\log(R-t_n^*)}$$

are bounded as $n \to \infty$. Hence

$$\frac{\log \mu(t_n^*)}{-\log(R-t_n^*)} \quad \text{and} \quad \frac{\log \lambda(s_n^*)}{-\log(R-s_n^*)}$$

are also bounded. Since $F(z)$ is not a rational function

$$\lim_{n\to\infty} \frac{\log M(s_n^*, F)}{-\log(R-s_n^*)} = \lim_{n\to\infty} \frac{\log M(t_n^*, F)}{-\log(R-t_n^*)} = \infty .$$

Therefore, $M(s_n^*, F) > \lambda(s_n^*)$ and $M(t_n^*, F) > \mu(t_n^*)$. But

$$M(s_n^*, F) < \delta(s_n^*) \leqslant \mu^*(s_n^*) = \mu(s_n^*) , \qquad M(t_n^*, F) < \delta(t_n^*) \leqslant \lambda^*(t_n^*) = \lambda(t_n^*) .$$

Now using the continuity of the maximum modulus $M(r, F)$ and the functions $\lambda(r)$ and $\mu(r)$, we get $M(r, F) = \mu(r)$ on one sequence tending to R and $M(r, F) = \lambda(r)$ on another such sequence. This completes the proof of the theorem.

Corollary. *Let $\lambda_1(r)$ and $\mu_1(r)$ be positive and continuous functions of r for $0 < R < r < \infty$ such that* $\log \lambda_1(r) \neq 0(-\log(r - R))$ *and* $\log \mu_1(r) \neq 0(-\log(r - R))$ *as $r \to R^+$. Then there exists a function $G(z)$ belonging to U_R^c $(0 < R < \infty)$ with nonnegative coefficients such that $M(r, G) = \lambda_1(r)$ on some sequence of values of r decreasing to R and $M(r, G) = \mu_1(r)$ on another such sequence.*

References.

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Summary.

Let U_R be the class of functions analytic in the disc D_R with centre as origin and radius $R(0 < R < \infty)$. We prove the existence of functions in the class U_R whose upper rate of growth is arbitrarily fast and simultaneously whose lower rate of growth is arbitrarily slow as measured by their maximum moduli. The existence of functions in the class U_R having arbitrarily prescribed rates of growth on two different although unspecified sequences tending to R is also established.

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