V. P. GUPTA and S. K. ANAND (*)

On the means of an entire Dirichlet series of order (R) zero. (**)  

1. - Introduction.

A Dirichlet series

\[ f(s) = \sum_{n=1}^{\infty} a_n \exp(s\lambda_n), \quad s = \sigma + it, \]

where \( 0 = \lambda_0 < \lambda_1 < \lambda_2 \ldots < \lambda_n \to \infty \) as \( n \to \infty \), which we assume to be absolutely convergent everywhere in the complex plane \( \mathbb{C} \), is bounded in any left strip and defines an entire function. The order of \( f(s) \) is defined as:

\[ \lim_{\sigma \to \infty} \sup \frac{\log \log M(\sigma)}{\sigma} = \varrho \quad (0 < \varrho < \infty), \]

where \( M(\sigma) = \sup \{|f(\sigma + it)|; -\infty < t < \infty\} \).

To have a more precise description of the growth relation for a class of entire Dirichlet series of order (R) zero, i.e. for which \( \varrho = 0 \), we use the notions of logarithmic order (R), \( q^* \), and the lower logarithmic order (R), \( \lambda^* \), as given by (see [1], [2])

\[ \lim_{\sigma \to \infty} \sup \frac{\log \log M(\sigma)}{\log \sigma} = \varrho^* \quad (1 < \varrho^* < q^* < \infty), \]

\[ \lim_{\sigma \to \infty} \inf \frac{\log \log M(\sigma)}{\log \sigma} = \lambda^* \quad (1 < \lambda^* < q^* < \infty). \]

(*) Indirizzo degli Autori: V. P. GUPTA, M.M.H. College, Ghaziabad (U.P.), India; S. K. ANAND, Faculty of Mathematics, University of Delhi, Delhi-110007, India,  
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Consider the following mean values of $|f(s)|$

$$I_d(\sigma) = \lim_{\varepsilon \to 0} \frac{1}{2\pi} \int_{-\varepsilon}^{\varepsilon} |f(\sigma + it)|^d dt \quad (0 < \delta < \infty),$$

$$m_{k,2}(\sigma) = \exp \left( \frac{k}{\sigma} \right) \int_{0}^{\infty} \exp \left( kx \right) I_d(x) dx \quad (0 < k < \infty).$$

Kamthan and Jain have obtained a number of growth relations regarding these means in $(I)_1, (I)_2, (3)_1$ for entire Dirichlet series of order $(R), \varrho (0 < \varrho < \infty)$. In this Note, our main object is to discuss certain properties of these means for functions of logarithmic order $(\beta), \varrho^\beta$, and lower logarithmic order $(\gamma), \lambda^\gamma$.

2. Theorem 1. Let $f(s)$ be an entire function represented by Dirichlet series of logarithmic order $\varrho^\beta$ and lower logarithmic order $\lambda^\gamma$. Then, for $1 < \delta < \infty$,

$$\lim_{\sigma \to \infty} \sup_{\sigma} \frac{\log \left( \frac{I_d^{(\beta)}(\sigma)}{I_d(\sigma)} \right)}{\log \sigma} = \frac{\varrho^\beta - 1}{\lambda^\gamma - 1} \quad (1 < \lambda^\gamma < \varrho^\beta < \infty)$$

where $I_d^{(\beta)}(\sigma) = I_d(\sigma, \gamma)$.

The proof of this theorem is based upon the following lemmas.

Lemma 1. For $0 < \delta < \infty$,

$$\lim_{\sigma \to \infty} \sup_{\sigma} \frac{\log \log I_d(\sigma)}{\log \sigma} = \varrho^\beta.$$

Proof. For $\eta > 0$, we have (see [2])

$$I_d(\sigma) < M(\sigma) < O(1) I_d(\sigma + \eta)$$

which, on using (1.1), proves the lemma.

Lemma 2 ([3]_1). For $\sigma > \sigma_0$ and $\delta > 1$

$$I_d^{(\beta)}(\sigma) > \frac{I_d(\sigma) \log I_d(\sigma)}{\sigma} (1 + O(1)).$$
Lemma 3. ([3],) With the usual notation for $I_3^{(1)}(\sigma)$, for all $\sigma > 0$ and $\eta > 0$,

$$ I_3^{(1)}(\sigma) < \frac{K}{\eta} I_3(\sigma + \eta) , $$

where $K$ is a constant.

Proof of Theorem 1. Since $\log I_3(\sigma)$ is a convex function with respect to $\sigma$ (see [3], lemma 5), we have

$$ \log I_3(\sigma) = \log I_3(\sigma_0) + \int_{\sigma_0}^{\sigma} \omega(x) \, dx \quad (\sigma > \sigma_0) , $$

where $\omega(x)$ is non-decreasing and almost continuous in the interval $(0, \infty)$; $\omega(x)$ tends to infinity with $x$. Therefore, for $\eta > 0$

$$ \log I_3(\sigma + \eta) = \log I_3(\sigma) + \int_{\sigma}^{\sigma + \eta} \omega(x) \, dx < \log I_3(\sigma) + \eta \omega(\sigma + \eta) $$

which, on using Lemma 3, gives

$$ \log I_3^{(1)}(\sigma) < \log I_3(\sigma) + \eta \omega(\sigma + \eta) - \log \eta + O(1) . $$

Choose $\eta = (\omega(\sigma + 2))^{-1}$.
Then, $\eta \omega(\sigma + \eta) < 1$, for all sufficient great values of $\sigma$. Hence

$$ \log I_3^{(1)}(\sigma) < \log I_3(\sigma) + \log \omega(\sigma + 2) + O(1) . $$

Also, from (2.3) and Lemma 1, it follows that

$$ \lim_{\sigma \to \infty} \frac{\sup \frac{\omega(\sigma)}{\log \sigma}}{\inf \frac{\log I_3(\sigma)}{\log \sigma}} = \frac{\sigma^* - 1}{\lambda^* - 1} . $$

This, from (2.5) (2.6), we find that

$$ \lim_{\sigma \to \infty} \frac{\sup \frac{\omega(\sigma)}{\log I_3^{(1)}(\sigma)/I_3(\sigma)}}{\inf \frac{\log I_3^{(1)}(\sigma)/I_3(\sigma)}} = \frac{\sigma^* - 1}{\lambda^* - 1} . $$

The reverse inequality is easily available from Lemma 1 and 2.
Theorem 2. Let \( f(\sigma) \) be an entire function represented by Dirichlet series of logarithmic order \( \rho^* \) and lower logarithmic order \( \lambda^* \). Then for \( \delta > 1, -1 < \kappa < \infty \)

\[
\lim_{\sigma \to \infty} \sup_{a \to 0} \frac{\log \left( \frac{m_{\delta, \kappa}(\sigma)}{m_{\delta, \kappa}(a)} \right)}{\log \sigma} = \frac{\rho^* - 1}{\lambda^* - 1},
\]

where \( m_{\delta, \kappa}(\sigma) = m_{\delta, \kappa}(\sigma, f^{(n)}) \).

We omit the proof, as it can easily be followed on the lines of the proof of Theorem 1.

3. – It is known that, for all entire functions,

\[
\lim_{\sigma \to \infty} \sup_{a \to 0} \frac{I_{\rho}(\sigma)}{m_{\delta, \kappa}(\sigma)}^{1/\rho} = e^\rho \\
(0 < \lambda < \rho < \infty).
\]

In particular, for entire functions of order \((R)\) zero, i.e. \( \rho = 0 \) we have

\[
(3.1) \quad \lim_{\sigma \to \infty} \frac{I_{\rho}(\sigma)}{m_{\delta, \kappa}(\sigma)}^{1/\rho} = 1.
\]

In what follows, we have a result for entire functions of order \((R)\) zero, which is more precise than (3.1), namely.

Theorem 3. Let \( f(\sigma) \) be an entire function of logarithmic order \( \rho^* \) and lower logarithmic order \( \lambda^* \). Then

\[
\lim_{\sigma \to \infty} \sup_{a \to 0} \frac{I_{\rho}(\sigma)}{m_{\delta, \kappa}(\sigma)}^{1/\rho} = \exp \left( \frac{\rho^* - 1}{\lambda^* - 1} \right) \\
(1 < \lambda^* < \rho^* < \infty).
\]

Before proving this theorem, we will firstly prove the following

Lemma 4.

\[
\lim_{\sigma \to \infty} \sup_{a \to 0} \frac{\log \log m_{\delta, \kappa}(a)}{\log \sigma} = \frac{\rho^*}{\lambda^*}.
\]

Proof. Lemma follows directly from Lemma 1 and the inequalities

\[
m_{\delta, \kappa}(\sigma) < \frac{I_{\rho}(\sigma)}{e} < m_{\delta, \kappa}(\sigma + \eta)(1 + o(1))^{-1} \\
(\eta > 0).
\]
Proof of Theorem 3. It is seen, from the definition of \( I_\sigma(\sigma) \) and \( m_{\sigma,1}(\sigma) \), that (see (1.1))

\[
\log m_{\sigma,1}(\sigma) = \log m_{\sigma,2}(\sigma_0) + \int_{\sigma_0}^{\sigma} \varphi(x) \, dx,
\]

where

\[
\varphi(x) = \frac{I_{\sigma}(x)}{m_{\sigma,2}(x)} - k
\]

(3.2)

is an increasing function of \( x \), for all large \( x \) (see (3.1), lemma 3). Thus, for all \( \sigma \geq \sigma_0 \)

\[
\log m_{\sigma,1}(\sigma) - \log m_{\sigma,1}(\sigma_0) \leq \varphi(\sigma)(\sigma - \sigma_0),
\]

which in view of Lemma 4, yields

\[
\lim_{\sigma \to \infty} \sup_{\sigma_0} \frac{\log \varphi(x)}{\log \sigma} > \frac{\sigma - 1}{\lambda^* - 1}. \tag{3.3}
\]

Again we have

\[
\log m_{\sigma,1}(2\sigma) > \int_{\sigma}^{\lambda^*} \varphi(x) \, dx > \sigma \varphi(\sigma),
\]

which again using Lemma 4, yields

\[
\lim_{\sigma \to \infty} \sup_{\sigma_0} \frac{\log \varphi(x)}{\log \sigma} < \frac{\sigma - 1}{\lambda^* - 1}. \tag{3.4}
\]

Hence, from (3.3) and (3.4), we get

\[
\lim_{\sigma \to \infty} \frac{\sup \varphi(x)}{\log \sigma} \leq \frac{\sigma - 1}{\lambda^* - 1}. \tag{3.5}
\]

The theorem now follows from (3.2) and (3.5).

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References.


Summary.

For an entire Dirichlet series $f(s) = \sum_{n=1}^{\infty} \alpha_n \exp(\alpha_n) (s = \sigma + it, \lambda_{n+1} \geq \lambda_n \to \infty$ with $n)$ of order (R) zero, the logarithmic order (R) $\xi^0$ and the lower logarithmic order (R) $\lambda^0$ have been defined as

$$\lim_{\sigma \to \infty} \sup \log \log M(a) = \frac{\xi^0}{\lambda^0} \quad (1 < \lambda^0 < \xi^0 < \infty),$$

where $M(a) = \text{Max} \{|f(\sigma + it)|: -\infty < t < \infty\}$. In this paper, certain properties of the mean values $I(\sigma)$ and $M(\sigma)$ of functions of logarithmic order (R)$\xi^0$ and lower logarithmic order (R)$\lambda^0$ have been obtained.