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# On the absolute Nörlund summability of ultraspherical series. (**)

1. – Let $\sum_{n=0}^{n=\infty} a_n$ be a given infinite series with the sequence of partial sums $\{s_n\}$. Let $\{p_n\}$ be a sequence of constants, real or complex and let us write:

$$P_n = p_0 + p_1 + \dots + p_n\,, \qquad P_{-1} = p_{-1} = 0\,, \quad P_n \neq 0\,.$$

Let

$$t_n = \frac{1}{P_n} \sum_{k=0}^{k=n} p_{n-k}\, s_k \tag{1.1}$$

define the sequence of Nörlund means [6] of the sequence $\{s_n\}$. The series $\sum a_n$ is said to be *absolutely summable* $(N, p_n)$, or *summable* $|N, p_n|$, if the sequence $\{t_n\}$ is of bounded-variation that is the series

$$\sum_n |t_n - t_{n-1}| \tag{1.2}$$

is convergent [5].

The Cesàro summability becomes a special case of the Nörlund summability [3], when

$$p_n = \binom{n+\alpha-1}{\alpha-1} = \frac{\Gamma(n+\alpha)}{\Gamma\alpha\,\Gamma(n+1)} \qquad (\alpha > 0)\,.$$

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Also when $p_n = 1/(n+1)$, the Nörlund summability is termed as the harmonic summability.

2. – The ultraspherical polynomials $P_n^{(\lambda)}(x)$ are defined by the following expansion

$$(1 - 2xt + t^2)^{-\lambda} = \sum_{n=0}^{\infty} t^n P_n^{(\lambda)}(x) \qquad (\lambda > 0)\,. \tag{2.1}$$

If $f(\theta, \varphi)$ be a function defined on the range $0 \leqslant \theta \leqslant \pi$ and $0 \leqslant \varphi \leqslant 2\pi$, the ultraspherical series corresponding to it on the sphere $S$ is

$$f(\theta, \varphi) \sim \frac{1}{2\pi} \sum_{n=0}^{\infty} (n+\lambda) \iint_S \frac{f(\theta', \varphi') P_n^{(\lambda)}(\cos\omega) \sin\theta' \mathrm{d}\theta' \mathrm{d}\varphi'}{[\sin^2\theta' \sin^2(\varphi - \varphi')]^{\frac{1}{2}-\lambda}}\,, \tag{2.2}$$

where

$$\cos\omega = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi - \varphi')\,.$$

The Laplace series is a particular case of this series for $\lambda = \frac{1}{2}$, while this reduce to trigonometric series in the limit as $\lambda \to 0$, because

$$\lim_{\lambda \to 0} \frac{1}{\lambda} P_n^{(\lambda)}(\cos\theta) = (2/n)\cos n\theta \qquad (n \geqslant 1)\,. \tag{2.3}$$

A generalised mean value of $f(\theta, \varphi)$ on the sphere has been defined by Kogbetliantz [**5**] in 1924 as follows:

$$f(\omega) = \frac{1}{2\pi(\sin\omega)^{2\lambda}} \int_{\sigma\omega} \frac{f(\theta', \varphi') \sin\theta' \mathrm{d}\theta' \mathrm{d}\varphi'}{[\sin^2\theta' \sin^2(\varphi - \varphi')]^{\frac{1}{2}-\lambda}}\,. \tag{2.4}$$

Where the integral is taken along the small circle whose centre is $(\theta, \varphi)$ on the sphere and whose curvilinear-radius is $\omega$.

It is assumed throughout that the function

$$f(\theta', \varphi')[\sin^2\theta' \sin^2(\varphi - \varphi')]^{\lambda - \frac{1}{2}} \tag{2.5}$$

is absolutely integrable $(L)$ over the sphere $S$.

The object of this paper is to obtain a simple direct theorem on the absolute Nörlund summability of the series (2.2). Writing

$$\varphi(\omega) = \frac{(\sin\omega)^{2\lambda-1}\Gamma\lambda\, f(\omega)}{2\,\Gamma\frac{1}{2}\,\Gamma(\frac{1}{2}+\lambda)},$$

we prove the following

Theorem. *Let* $\{p_n\}$ *be a non negative non increasing sequence such that for* $1/2 \leqslant \lambda < 1$

$$\text{(i)}\quad \sum_n \frac{n^{\lambda-1}}{P_n} < \infty, \qquad \text{(ii)}\quad \int_0^\pi \frac{|d\varphi(\omega)|}{(\sin\omega)^\lambda} < \infty.$$

*Then the series* (2.2) *is summable* $|N, p_n|$ *at the point* $(\theta, \varphi)$ *of the sphere.*

**3.** – We require the following lemmas for the proof of our theorem.

Lemma 1 (Ahmad [**1**]). *If* $p_0 > 0$ *and* $p_n$ *is non negative and non increasing sequence, then* for $\nu \geqslant 1$

$$\sum_{n=\nu}^{\infty} \frac{p_n p_{n-\nu}}{P_n P_{n-1}} \leqslant \frac{C}{\nu}, \tag{3.1}$$

$$\sum_{n=\nu}^{\infty} \frac{p_n(p_n - p_{n-\nu})}{P_n - P_{n-1})} \leqslant C, \tag{3.2}$$

$$\sum_{n=\nu}^{\infty} \frac{|\Delta_n p_{n-\nu-1}|}{P_{n-1}} \leqslant \frac{C}{P_\nu} + \frac{C}{\gamma}, \tag{3.3}$$

$$\sum_{n=\nu}^{\infty} \frac{(p_{n-\nu} - p_n)}{P_{n-1}} \leqslant C, \tag{3.4}$$

*C denote an absolute costant.*

Lemma 2 (Kogbetliantz [**4**]). *If*

$$\frac{\pi}{n+1} \leqslant \theta \leqslant \pi - \frac{\pi}{n+1}, \qquad \lambda > 0,$$

*then*

$$P_n^{(\lambda)}(\cos\theta) = 2\,\frac{A_n^{\lambda-1}\cos[(n+\lambda)\theta - \lambda\pi/2]}{(2\sin\theta)^\lambda} + \frac{k}{(n+1)^{2-\lambda}(\sin\theta)^{\lambda+1}} \tag{3.5}$$

*k a fixed costant.*

Lemma 3 (Kogbetliantz [3]). *If* $0 \leqslant \theta \leqslant \pi$, $\lambda > 0$, $n = 0, 1, 2, \ldots$, *then*

$$|P_n^{(\lambda)}(\cos\theta)| \leqslant 2(\sin\theta)^{-\lambda} A_n^{\lambda-1}, \tag{3.6}$$

*where*

$$A_n^q = \binom{n+q}{q} \sim n^q.$$

Lemma 4. *If* $1/2 \leqslant \lambda < 1$ *and* $\int\limits_0^\pi \frac{|\mathrm{d}\varphi(\omega)|}{(\sin\omega)^\lambda} < \infty$,

*then*

$$S_n = O(n^{\lambda-1}). \tag{3.7}$$

Proof. The result of the lemma has been given by Gupta [2] and we reproduce below some relevant formulae from which this estimate may be obtained.

The $n$th partial sum of the series (2.2) is given by

$$S_n = \frac{\Gamma\lambda}{2\,\Gamma\frac{1}{2}\Gamma(\frac{1}{2}+\lambda)} \int\limits_0^\pi f(\omega)\left[\frac{\mathrm{d}}{\mathrm{d}x}\,\{P_{n+1}^{(\lambda)}(x) + P_n^{(\lambda)}(x)\}\right]_{x=\cos\omega} (\sin\omega)^{2\lambda}\,\mathrm{d}\omega$$

$$= \int\limits_0^\pi \varphi(\omega)\,\frac{\mathrm{d}}{\mathrm{d}\omega}\,\{P_{n+1}^{(\lambda)}(\cos\omega) + P_n^{(\lambda)}(\cos\omega)\}\,\mathrm{d}\omega$$

$$= \varphi(\pi)[P_{n+1}^{(\lambda)}(-1) + P_n^{(\lambda)}(-1)] - \int\limits_0^\pi \{P_{n+1}^{(\lambda)}(\cos\omega) + P_n^{(\lambda)}(\cos\omega)\}\,\mathrm{d}\varphi(\omega) = U_2 - U_1.$$

Since

$$P_n^{(\lambda)}(1) = \binom{n+2\lambda-1}{n},$$

it is clear that

$$U_1 = (-1)^n\,\varphi(\pi)\,\frac{\Gamma(n+2\lambda)}{\Gamma(n+2)\,\Gamma(2\lambda)}(1-2\lambda) \sim A n^{2\lambda-2},$$

and

$$U_2 = \int\limits_0^\pi \{P_{n+1}^{(\lambda)}(\cos\omega) + P_n^{\lambda}(\cos\omega)\}\,\mathrm{d}\varphi(\omega) = \int\limits_0^{\pi/n+1} + \int\limits_{\pi/n+1}^{\pi-\pi/n+1} + \int\limits_{\pi-\pi/n+1}^{\pi} = I_1 + I_2 + I_3.$$

Using (3.6) in $I_1, I_3$ and (3.5) in $I_2$ it can be easily seen that $U_2 = O(n^{\lambda-1})$. Hence $S_n = O(n^{\lambda-1})$.

**4.** – Proof of the Theorem. Let $T_n$ denote the $n$-th Nörlund mean of the series (2.2). Then by definition

$$T_n - T_{n-1} = \sum_{\nu=0}^{\nu=n} \frac{P_{n-\nu}}{P_n} a_\nu - \sum_{\nu=0}^{\nu=n-1} \frac{P_{n-1-\nu}}{P_{n-1}} a_\nu = \sum_{\nu=1}^{\nu=n} \left( \frac{P_{n-\nu}}{P_n} - \frac{P_{n-1-\nu}}{P_{n-1}} \right) a_\nu$$

$$= \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{\nu=n} (P_n - P_{n-\nu}) a_\nu + \frac{1}{P_{n-1}} \sum_{\nu=1}^{\nu=n} (p_{n-\nu} - p_n) a_\nu$$

$$= \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{\nu=n-1} \Delta_\nu\{(P_n - P_{n-\nu})\} S_\nu + (P_n - P_0) \frac{p_n}{P_n P_{n-1}} S_n +$$

$$+ \frac{1}{P_{n-1}} \sum_{\nu=1}^{\nu=n-1} \Delta_\nu\{(p_{n-\nu} - p_n)\} S_\nu + \frac{1}{P_{n-1}} (p_0 - p_n) S_n$$

$$= \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{\nu=n} \Delta_\nu \{(P_n - P_{n-\nu})\} S_\nu + \frac{1}{P_{n-1}} \sum_{\nu=1}^{\nu=n} \Delta_\nu (p_{n-\nu} - p_n) S_\nu ,$$

and therefore

$$|T_n - T_{n-1}| \leqslant \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{\nu=n} |\Delta_\nu\{(P_n - P_{n-\nu})\}| \, |S_\nu| + \frac{1}{P_{n-1}} \sum_{\nu=1}^{\nu=n} |\Delta_\nu\{p_{n-\nu} - p_n\}| \, |S_\nu| .$$

Hence for establishing the theorem we have to prove that

$$\sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{\nu=n} |\Delta_\nu\{P_n - P_{n-\nu}\}| \, |S_\nu| < \infty \tag{4.1}$$

and

$$\sum_{n=1}^{n=\infty} \frac{1}{P_{n-1}} \sum_{\nu=1}^{\nu=n} |\Delta_\nu\{p_{n-\nu} - p_n\}| \, |S_\nu| < \infty . \tag{4.2}$$

Now

$$\sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{\nu=n} |\Delta_\nu\{P_n - P_{n-\nu}\}| \, |S_\nu| = \sum_{n=1}^{\infty} \frac{p_n}{P_n P_{n-1}} \sum_{\nu=1}^{\nu=n} |p_{n-\nu}| \, |S_\nu| =$$

$$= \sum_{\nu=1}^{\nu=\infty} |S_\nu| \sum_{n=\nu}^{\infty} \frac{p_n p_{n-\nu}}{P_n P_{n-1}} = O(1) \sum_{\nu=1}^{\infty} \frac{|S_\nu|}{\nu} = O(1) \sum_{\nu=1}^{\infty} \frac{\nu^{\lambda-1}}{\nu} = O(1) ,$$

by the application of the estimates (3.1), (3.7) and using the fact that $1/2 \leqslant \lambda < 1$. This complites the proof of the estimate in (4.1).

Now we proceed to estimate (4.2) we have

$$\sum_{n=1}^{\infty} \frac{1}{P_{n-1}} \sum_{\nu=1}^{\nu=n} |\Delta_\nu\{p_{n-\nu} - p_n\}| \, |S_\nu| = \sum_{n=1}^{\infty} \frac{1}{P_{n-1}} \sum_{\nu=1}^{\nu=n} \{|\Delta p_{n-\nu}|\} \, |S_\nu| =$$

$$= \sum_{\nu=1}^{\infty} |S_\nu| \sum_{n=\nu}^{\infty} \left\{ \frac{|\Delta_n p_{n-1-\nu}|}{P_{n-1}} \right\} =$$

$$= O(1) \sum_{\nu=1}^{\infty} \left\{ \frac{|S_\nu|}{P_\nu} + \frac{|S_\nu|}{\nu} \right\} = O(1) \left[ \sum_{\nu=1}^{\infty} \frac{|S_\nu|}{P_\nu} + \sum_{\nu=1}^{\infty} \frac{|S_\nu|}{\nu} \right] =$$

$$= O(1) \left[ \sum_{\nu=1}^{\infty} \frac{\nu^{\lambda-1}}{P_\nu} + \sum_{\nu=1}^{\infty} \frac{\nu^{\lambda-1}}{\nu} \right] = O(1),$$

by virtue of the estimates (3.3), (3.7) and hypothesis of the theorem. This completes the proof of (4.2) and combining (4.1) and (4.2) the theorem is established.

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## References.

[1] Z. U. Ahmad, *On the absolute Nörlund summability factors of a Fourier series*, Riv. Mat. Univ. Parma (2) **7** (1966), 157-169.

[2] D. P. Gupta, *Absolute Cesàro summability of the ultraspherical series*, Vijnana parisnad Anusandhan Patrika (2) **2** (1968), 91-95.

[3] G. H. Hardy, *Divergent series*, Oxford 1949.

[4] E. Kogbetliantz, *Sur la summabilité de la série ultrasphérique à l'intérieur de l'intervalle* (— 1, +1) *par la méthode des moyennes arithmétique*, Bull. Soc. Math. France **51** (1923), 244-295.

[5] F. M. Mears, *Some multiplication theorems for the Nörlund means*, Bull. Amer. Math. Soc. **41** (1935), 875-880.

[6] N. G. Nörlund, *Sur une application des permutables*, Lunds Universitets Arsskrift (2) **16** (1919), 1-10.

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