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Linear homeomorphisms in the space of analytic functions in bicylinders. (**)

1. - Introduction and terminology.

Let $\mathscr{C}$ denote the usual complex plane and $\mathscr{C}^2$, the cartesian product of $\mathscr{C}$ with itself, equipped with the usual product topology. Let χ be the class of all functions $f: \mathscr{C}^2 \to \mathscr{C}$, where each f is analytic in $P(R_1, R_2) = \{(z_1, z_2): |z_1| < R_1, |z_2| < R_2\}$ for arbitrary but fixed R_1 and R_2 such that $0 < R_1$, $R_2 \leqslant +\infty$. Let us now recall the definitions and notations and the various equivalent topologies introduced on the space χ in our earlier paper [**3**]. We write $\delta_{mn}(z_1, z_2) = z_1^m z_2^n$. Then each $f \in \chi$ is uniquely representable by the infinite series

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{mn} \delta_{mn} ,$$

which is uniformly and absolutely convergent on compacta in bicylinder $P(R_1, R_2)$. We denote by $\mathscr{F}$ the topology on χ of uniform convergence on compacta in $\mathscr{C}^2$ and thus χ, equipped with $\mathscr{F}$, becomes a Fréchet space. For any function f in χ and $0 < r_1 < R_1$ and $0 < r_2 < R_2$, we employ the following notations

$$M(f; r_1, r_2) = \max_{|z_1| \leqslant r_1, |z_2| \leqslant r_2} |f(z_1, z_2)|$$

and [1]

$$\|f; r_1, r_2\| = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} |a_{mn}| r_1^m r_2^n . \tag{1.1}$$

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(1) In [**3**], we have used the notation $p(f; r_1, r_2)$ for $\|f; r_1, r_2\|$.

By the application of Cauchy's fundamental inequality for several complex variables ([2], p. 49) it can be easily seen that

$$(1.2)\qquad \begin{cases} M(f; r_1, r_2) \leqslant \|f; r_1, r_2\| \leqslant \dfrac{\varrho_1}{(\varrho_1 - r_1)} \dfrac{\varrho_2}{(\varrho_2 - r_2)} M(f; \varrho_1, \varrho_2) \\ r_1 < \varrho_1 < R_1;\ r_2 < \varrho_2 < R_2 . \end{cases}$$

In [3] it has been proved that the toplogy $\mathscr{F}$ is the same as generated by $\{M(f; r_1^{(n)}, r_2^{(n)})\colon n \geqslant 1\}$ or $\{\|f; r_1^{(n)}, r_2^{(n)}\|, n \geqslant 1\}$, where $0 < r_i^{(1)} < r_i^{(2)} < \dots$ and $r_i^{(n)} \to \to R_i$ as $n \to \infty$; $i = 1, 2$. Throughout this Note we write χ to mean that χ is equipped with $\mathscr{F}$. For fixed r_1 and r_2 such that $r_1 < R_1$ and $r_2 < R_2$, we denote by $\chi(r_1, r_2)$, the space χ equipped with the topology generated by the norm $\|\dots; r_1, r_2\|$.

Now, let us consider a double sequence $\{\alpha_{mn}\}$ in the space χ. We call a sequence $\{\alpha_{mn}\}$ to be *linearly independent* if

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{mn} \alpha_{mn} = 0 \Rightarrow a_{mn} = 0 \qquad (m = 0, 1, \dots;\ n = 0, 1, 2, \dots)$$

for all sequences $\{a_{mn}\}$ of complex numbers for which the series converges. A sequence $\{\alpha_{mn}\}$ is said to *span* a subspace χ_0 of χ if χ_0 consists of all linear combinations of the form

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{mn} \alpha_{mn} ,$$

where $\{a_{mn}\}$ is any sequence of complex numbers for which the series converges in χ. We call $\{\alpha_{mn}\}$ to be *basis* of a subspace χ_0 of χ if it is linearly independent and spans χ_0. A double sequence $\{\alpha_{mn}\colon m, n \geqslant 0\} \subset \chi$ which is also a basis for a subspace χ_0 of χ is said to be a *proper base* for χ_0 if for all sequences $\{a_{mn}\}$ of complex numbers

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{mn} \delta_{mn} \text{ converges} \Leftrightarrow \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{mn} \alpha_{mn} \text{ converges} .$$

We have shown in ([3], Theorem 2.3′) that if a basis $\{\alpha_{mn}\}$ is a proper basis, then

$$(\alpha)\qquad \limsup_{m+n \to \infty} \left[\frac{M(\alpha_{mn}; r_1, r_2)}{R_1^m R_2^n}\right]^{1/(m+n)} < 1$$

holds for all $r_1 < R_1$, $r_2 < R_2$.

The main object of this paper is to characterize the linear homeomorphic mappings from χ onto closed subspaces of χ which carry $\{\delta_{mn}\}$ into a proper basis.

2. - Characterizations.

For the proof of the main result, we shall need the following two lemmas.

Lemma 1. *Let T be a linear mapping of χ into itself. Then T is continuous if and only if for each $r_1 < R_1$ and $r_2 < R_2$, there correspond a $\varrho_1 < R_1$ and a $\varrho_2 < R_2$ such that T is a continuous map from $\chi(\varrho_1, \varrho_2)$ into $\chi(r_1, r_2)$.*

Proof. Let T be a continuous map, but for some $r_1 < R_1$ and $r_2 < R_2$, it does not map any $\chi(\varrho_1, \varrho_2)$ continuously into $\chi(r_1, r_2)$. This implies that we can find positive increasing sequences $\{\varrho_1^{(n)}\}$ and $\{\varrho_2^{(n)}\}$ such that $\varrho_1^{(n)} \to R_1$ and $\varrho_2^{(n)} \to R_2$ as $n \to \infty$, and also a sequence $\{f_n\}$ of functions in χ such that

$$\|f_n;\, \varrho_1^{(n)}, \varrho_2^{(n)}\| \to 0\,, \tag{2.1}$$

$$\|Tf_n;\, r_1, r_2\| = 1\,. \tag{2.2}$$

From (2.1), it clearly follows that $\{f_n\}$ converges to zero in χ; but by (2.2) $\{Tf_n\}$ can not converge to 0 in χ. This contradicts that T is continuous map from χ to χ and the necessary part is proved.

To prove the converse, we are given that for each $r_1 < R_1$ and $r_2 < R_2$, we can find $\varrho_1 < R_1$ and $\varrho_2 < R_2$ such that T maps $\chi(\varrho_1, \varrho_2)$ continuously into $\chi(r_1, r_2)$. Let $\{f_n\}$ be a sequence in χ such that $f_n \to 0$ in the topology of uniform convergence on compacta in $\varrho(R_1, R_2)$

$$\|f_n;\, \varrho_1, \varrho_2\| \to 0 \quad \text{for all} \quad \varrho_1 < R_1 \quad \text{and} \quad \varrho_2 < R_2\,.$$

Now, this result and our hypothesis imply that

$$\|Tf_n;\, r_1, r_2\| \to 0 \qquad \text{for each } r_1 < R_1 \qquad \text{and each } r_2 < R_2\,,$$

i.e. $\{Tf_n\}$ converges to zero uniformly on compact sets. Hence, T is continuous. This completes the proof.

Lemma 2. *If T is a continuous linear mapping of χ into itself and $\{\alpha_{mn}\}$ a sequence of functions in χ with $T\delta_{mn} = \alpha_{mn}$ $(m = 0, 1, \ldots; n = 0, 1, \ldots)$, then*

$$(\alpha) \qquad \limsup_{m+n\to\infty} \left[\frac{M(\alpha_{mn}; r_1, r_2)}{R_1^m R_2^n}\right]^{1/(m+n)} < 1$$

holds for all $r_1 < R_1$ and all $r_2 < R_2$. Conversely, if $\{\alpha_{mn}\}$ is a sequence of functions in χ for which condition (α) holds, then there exists a continuous linear mapping T of χ into itself such that $T\delta_{mn} = \alpha_{mn}$ $(m, n = 0, 1, 2, \ldots)$.

Proof. For proving the necessary part, we fix $r_1 < R_1$ and $r_2 < R_2$. Then, by Lemma 1, we can find $\varrho_1 < R_1$ and $\varrho_2 < R_2$ such that T maps $\chi(\varrho_1, \varrho_2)$ continuously into $\chi(r_1, r_2)$.
$\Rightarrow$ There exists a constant K such that

$$\|\alpha_{mn}; r_1, r_2\| = \|T\delta_{mn}; r_1, r_2\| \leqslant K\|\delta_{mn}; \varrho_1, \varrho_2\| = K\varrho_1^m \varrho_2^n \qquad (m, n = 0, 1, 2, \ldots),$$

$$\Rightarrow \left[\frac{\|\alpha_{mn}; r_1, r_2\|}{R_1^m R_2^n}\right]^{1/(m+n)} \leqslant K^{1/(m+n)} \left(\frac{\varrho_1}{R_1}\right)^{m/(m+n)} \left(\frac{\varrho_2}{R_2}\right)^{n/(m+n)}$$

$$(m, n = 0, 1, 2, \ldots, m + n \neq 0).$$

Choose $\alpha = \max\{\varrho_1/R_1, \varrho_2/R_2\}$. Then obviously $\alpha < 1$ and we get

$$\left[\frac{\|\alpha_{mn}; r_1, r_2\|}{R_1^m R_2^n}\right]^{1/(m+n)} \leqslant K^{1/(m+n)}\alpha, \qquad \limsup_{m+n\to\infty} \left[\frac{\|\alpha_{mn}; r_1, r_2\|}{R_1^m R_2^n}\right]^{1/(m+n)} \leqslant \alpha < 1.$$

Thus (α) holds.

Conversely, suppose that (α) holds and fix $r_1 < R_1$ and $r_2 < R_2$ arbitrarily. Choose $\varrho < 1$ such that

$$\limsup_{m+n\to\infty} \left[\frac{\|\alpha_{mn}; r_1, r_2\|}{R_1^m R_2^n}\right]^{1/(m+n)} < \varrho < 1.$$

[This follows from the inequality (1.2)].
$\Rightarrow$ $\|\alpha_{mn}; r_1, r_2\| \leqslant \varrho^{m+n} R_1^m R_2^n$, for sufficiently large $m + n$.

Put $\varrho_1 = \varrho R_1$ and $\varrho_2 = \varrho R_2$. Then $\varrho_1 < R_1$ and $\varrho_2 < R_2$. We get

$$\|\alpha_{mn}; r_1, r_2\| \leqslant \varrho_1^m \varrho_2^n \qquad \text{for sufficiently large } m + n.$$

We can now find a constant K such that

$$(2.3) \qquad \|\alpha_{mn}; r_1, r_2\| \leqslant K\varrho_1^m \varrho_2^n \quad \text{for} \quad m = 0, 1, 2, \ldots, \quad n = 0, 1, 2, \ldots.$$

Since any element $f \in \chi$ can be uniquely represented by

$$f(z_1, z_2) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{mn} \delta_{mn} ,$$

where the series on the right hand side converges in the topology of χ, we define a linear transformation T on χ by

$$T(f) = T\left(\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{mn} \delta_{mn}\right) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{mn} \alpha_{mn} .$$

The convergence of the series $\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{mn} \alpha_{mn}$ is being assured by (2.3) and the convergence (2) of the series $\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{mn} \delta_{mn}$ in χ.

Moreover, from (2.3), it follows that

$$\| T\delta_{mn}; r_1, r_2 \| \leqslant K \| \delta_{mn}; \varrho_1, \varrho_2 \| \Rightarrow$$
$$\Rightarrow \| Tf; r_1, r_2 \| \leqslant K \| f; \varrho_1, \varrho_2 \| \quad \text{for} \quad f \in \chi .$$

Thus, T is a continuous linear map from $\chi(\varrho_1, \varrho_2)$ into $\chi(r_1, r_2)$. Using Lemma 1, we conclude that T is continuous map from χ into itself. This completes the proof of the lemma.

Now, we come to prove the main result of this Note, which is contained in the following theorem:

Theorem. *If T is a linear homeomorphic mapping of χ into itself, then $\{T\delta_{mn}\}$ is a proper basis in some closed subspace χ_0 of χ. Conversely, if $\{\alpha_{mn}\}$ is a proper basis in a closed subspace χ_0 of χ, then there exists a linear homeomorphic mapping T of χ into χ_0 such that $T\delta_{mn} = \alpha_{mn}$.*

Proof. Let T be a linear homeomorphism of χ into itself, and let χ_0 be the range of T. Then χ_0 is closed, for if we consider a function $f_0 \in \bar{\chi}_0$, we can find a net $\{f_\lambda\} \subset \chi_0$ such that $f_\lambda \to f_0$ in χ_0. Since T^{-1} is continuous on χ_0, it follows that $T^{-1}(f_\lambda) \to T^{-1}(f_0)$ in χ.

Since χ is complete, $T^{-1}(f_0) \in \chi \Rightarrow f_0 \in T(\chi) = \chi_0 \Rightarrow \chi_0$ is closed.

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(2) Here the convergence of $\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{mn} \alpha_{mn}$ is equivalent to the $\limsup_{m+n\to\infty} [|c_{mn}| R_1^m R_2^n]^{1/(m+n)} \leqslant 1$ (see [2]).

Thus χ_0 is a closed subspace of χ. Now we want to show that the sequence $\{\alpha_{mn}\}$ defined by $\alpha_{mn} = T\delta_{mn}$ is a proper basis in χ_0. Suppose $f \in \chi_0$. Then $T^{-1}f$ belongs to χ and so

$$T^{-1}f = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{mn}\delta_{mn} .$$

By continuity and linearity of T, it follows that

$$f = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{mn}\alpha_{mn} \Rightarrow \{\alpha_{mn}\} \text{ spans } \chi_0 .$$

Using Lemma 2 and a result in [**3**] (Theorem 2.1), it becomes clear that $\{\alpha_{mn}\}$ is a proper basis in χ_0 because f and $T^{-1}f$ have the same coefficient sequences relative to $\{\alpha_{mn}\}$ and $\{\delta_{mn}\}$ respectively.

For the converse, let χ_0 be a closed subspace of χ having a proper basis $\{\alpha_{mn}\}$. Then by theorem 2.3′ [3], condition (α) holds. By Lemma 2, there exists a continuous linear mapping T of χ into itself such that $T\delta_{mn} = \alpha_{mn}$ ($m = 0, 1, 2, \ldots; n = 0, 1, 2, \ldots$).

For any $\varphi \in \chi$, such that $\varphi = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{mn}\delta_{mn}$, we infer from the linearity and continuity of T that $T\varphi = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{mn}\alpha_{mn}$.

Therefore, T is one-to-one map of χ. Moreover it is onto because for any $\varphi \in \chi_0$, we have

$$\varphi = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{mn}\alpha_{mn} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{mn}T(\delta_{mn}) .$$

By continuity and linearily of T, it follows that

$$\varphi = T\Big(\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{mn}\delta_{mn}\Big) \qquad \text{where} \qquad \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{mn}\delta_{mn} \in \chi ,$$

since $\{\alpha_{mn}\}$ is a proper base. Thuse T is a one-to-one map of χ onto χ_0. Now χ_0, being a closed subspace of χ, is complete and hence by a theorem of Banach ([**1**], p. 41, Theorem 5), T is a homeomorphism.

This completes the proof of the theorem. By the composition of mappings in the usual fashion, following result follows:

Corollary. *If $\{\alpha_{mn}\}$ and $\{\beta_{mn}\}$ are proper bases for closed subspaces $\mathscr{P}$ and Q, respectively, of χ, then there exists a linear homeomorphic mapping T*

of $\mathscr{P}$ *onto* Q *such that* $T\alpha_{mn} = \beta_{mn}$ $(m, n = 0, 1, 2, \ldots)$. *Conversely, let* T *be a linear homeomorphic mapping of a closed subspace* $\mathscr{P}$ *of* χ *onto a closed subspace* Q *of* χ *and let* $\{\alpha_{mn}\}$ *be a proper basis in* $\mathscr{P}$, *then the sequence* $\{T\alpha_{mn}\}$ *is a proper basis in* Q.

References.

[1] S. Banach, *Théorie des opérations linéaires*, Hafner Publ. Co. N. Y. 1932.

[2] B. A. Fuks, *Introduction to the theory of analytic functions of several complex variables*, AMS Translations **8**, Rhode Island (1963).

[3] P. K. Kamthan and M. Gupta, *Analytic functions in bicylinders* (to appear in Ind J. Pure Appl. Math.).

Summary.

Essentially this paper deals with the linear homeomorphisms obtained as a consequence of equivalent proper bases in the space of analytic functions in bicylinders equipped with the natural open-compact Fréchet topology.

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