Logarithmic proximate order and geometric means of an entire function of order zero. (***)

1. - Introduction.

For a non-constant entire function \( f(z) \) of order zero, the \( L \)-order (logarithmic order), \( \varphi^* \), and the lower \( L \)-order, \( \lambda^* \), are given as [8]:

\[
\lim_{r \to \infty} \sup_{|z|=r} \frac{\log \log M(r, f)}{\log \log r} = \varphi^* = \lambda^* \quad (1 < \lambda^* < \varphi^* < \infty),
\]

where \( M(r, f) = \max_{|z|=r} |f(z)| \).

Let us define the following geometric means of \( f(z) \) for \( 0 < k < \infty \),

\[
G(r) = \exp \left\{ \frac{1}{2\pi} \int_{\theta}^{2\pi} \log |f(r \exp (i\theta))| \, d\theta \right\},
\]

\[
g_k(r) = \exp \left\{ \frac{k + 1}{r^{k+1}} \int_{0}^{r} x^* \log G(x) \, dx \right\},
\]

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(***) The work of P. K. Jain has been supported partially by the University Grants Commission, India. -- Ricevuto: 21-III-1972.
and

\[
\phi_k(r) = \exp \left\{ \frac{k + 1}{(\log r)^{k+1}} \int_1^r (\log x)^k \log G(x) \frac{dx}{x} \right\}.
\]

The mean value (1.2) was introduced by Kamthan [4] and a number of results regarding its growth with respect to \(G(r)\) and other auxiliary functions for an entire function of order \(\varrho\) were obtained in [2], [2], [4], [5]. In a recent paper [3], we have introduced a new geometric mean \(\phi_k^*(r)\) as defined in (1.3), and various relations involving the comparative growths of \(G(r)\), \(g_k(r)\) and \(\phi_k^*(r)\) relative to each other for an entire function of order zero have been established. It has been noted therein that the \(L\)-orders and the lower \(L\)-orders of the logarithms of these means are the same. Besides, the differences in the results regarding the growths of the pairs \((G(r), g_k(r))\) and \((G(r), \phi_k^*(r))\) have also been observed. The object of this paper is to continue a similar type of study by introducing \(L\)-proximate orders and thereby finding out the growth of \(G(r)\) and \(\phi_k^*(r)\). In section 2, we discuss certain preliminaries, whereas the remaining sections are devoted to our main results.

2. - Preliminaries.

It is assumed (throughout) that \(f(z)\) is a non-constant entire function of order zero. For these functions, we have

\[
\phi_1 = \text{g.l.b.} \left\{ x: x > 0 \text{ and } \sum_{n=1}^{\infty} r_n^{-x} < \infty \right\} = 0,
\]

where \(\{r_n\}_{n=1}^{\infty}\) denotes the sequence of the moduli of the zeros of \(f(z)\). To have a more precise description of the distribution of the zeros of such functions, we define a number \(\phi_1^*\) as

\[
\phi_1^* = \text{g.l.b.} \left\{ x: x > 0 \text{ and } \sum_{n=1}^{\infty} (\log r_n)^{-x} < \infty \right\},
\]

and call it the \(L\)-convergence (logarithmic convergence) exponent of the zeros of \(f(z)\) in analogy with \(\phi_1\), the convergence exponent of the zeros. Recently, the authors have proved that [3]:

\[
\limsup_{r \to \infty} \frac{\log n(r)}{\log \log r} = \phi_1^* \quad (0 < \phi_1^* < \infty),
\]
where \( n(r) \) is the number of the zeros of \( f(z) \) in the disc \( |z| < r \), and that \( q^* = q_1^* + 1 \). Also, we denote the limit inferior in (2.1) by \( \lambda_1^* \) and name it the lower \( L \)-convergence exponent of the zeros of \( f(z) \) in analogy with \( \lambda_1 \), the lower convergence exponent of the zeros, i.e.

\[
\liminf_{r \to \infty} \frac{\log n(r)}{\log \log r} = \lambda_1^* \quad (0 < \lambda_1^* < \infty).
\]

Also let

\[
N(r) = \frac{n(x)}{x} \, dx,
\]

where it is assumed, without any loss of generality, that \( n(r) = 0 \) for \( r < 1 \).

We define \( \mu(r) \), a real-valued function, to be a \( L \)-proximate (logarithmic proximate) order if it satisfies the following conditions:

(i) \( \mu(r) \) is continuous and differentiable in adjacent intervals for \( r > r_0 \),

(ii) \( \lim_{r \to \infty} \mu(r) = \mu \quad (0 < \mu < \infty) \),

and

(iii) \( \lim_{r \to \infty} r \cdot \mu'(r) \cdot \log r \cdot \log \log r = 0 \),

where \( \mu'(r) \) is either the right or the left-hand derivative at points where they are different.

We state below the existence theorem for the \( L \)-proximate order which can be easily proved on the lines of Levin ([6] p. 35):

**Theorem A.** If \( F(r) \) is any function that is positive for \( r > 1 \) and satisfies the conditions:

\[
\mu = \limsup_{r \to \infty} \frac{\log F(r)}{\log \log r} < \infty \quad (\mu > 0),
\]

then \( L \)-proximate order \( \mu(r) \) can be chosen so that

(iv) \( F(r) < (\log r)^{\mu(r)} \)

for \( r > r_0 \), and

(v) \( F(r) = (\log r)^{\mu(r)} \)

for a sequence \( r_n \) (\( n = 1, 2, 3, \ldots \)) of values of \( r \) tending to infinity.
Further, if $\nu$ ($0 < \nu < \infty$) be the limit inferior in (2.4), then following the lines of Shah [7] it is easy to prove the existence of lower $L$-proximate order $\nu(r)$ having the following conditions:

(i) $\nu(r)$ is real, continuous and differentiable in adjacent intervals for $r \geq r_0$,

(ii) $\lim_{r \to \infty} \nu(r) = \nu$ \hspace{1cm} ($0 < \nu < \infty$),

(iii) $\lim_{r \to \infty} r \cdot \nu'(r) \cdot \log r \cdot \log \log r = 0$,

where $\nu'(r)$ is the right-hand or left-hand derivative where the two differ,

(iv) $F(r) \geq (\log r)^{\nu(r)}$, for $r \geq r_0$,

and

(v) $F(r) = (\log r)^{\nu(r)}$,

for a sequence $r_m$ ($m = 1, 2, 3, \ldots$) of values of $r$ tending to infinity.

Now, computing exactly on the lines of Levin ([6], pp 33-35) one can deduce that:

(a) $(\log r)^{\mu(r)}$ is a monotone increasing function of $r$, for $r \geq r_0$, $\mu > 0$;

(b) for $r \to \infty$ and $0 < a < b < b < \infty$, the asymptotic inequality

$$(1 - \varepsilon)k^n (\log r)^{\mu(r)} < (\log r)^{\mu(r)} < (1 + \varepsilon)k^n (\log r)^{\mu(r)}$$

holds uniformly in $k$;

(c) for $p < \mu + 1$, \hspace{1cm} $\int_{r_0}^{\infty} (\log t)^{\mu(t) - p} \frac{dt}{t} \sim \frac{(\log r)^{\mu(r) + 1 - p}}{(\mu + 1 - p)}$

and

(d) for $p > \mu + 1$, \hspace{1cm} $\int_{r_0}^{\infty} (\log t)^{\mu(t) - p} \frac{dt}{t} \sim \frac{(\log r)^{\mu(r) + 1 - p}}{(p - \mu - 1)}$.

Also, following Singh and Dwivedi [9], we can easily obtain the various properties for lower $L$-proximate order $\nu(r)$ analogous to (a)-(d) of $\mu(r)$.
3. - Comparative growth of \( \log G(r) \) and \( \log g_k^*(r) \).

**Theorem 3.1.** If \( f(z) \) be an entire function of \( L \)-convergence exponent \( \varrho_1^* \) \((0 < \varrho_1^* < \infty)\) and lower \( L \)-convergence exponent \( \lambda_1^* \) \((0 < \lambda_1^* < \infty)\), then

\[
\liminf_{r \to \infty} \frac{\log g_k^*(r)}{\log G(r)} \leq \frac{k + 1}{\varrho_1^* + k + 2},
\]

and

\[
\limsup_{r \to \infty} \frac{\log g_k^*(r)}{\log G(r)} \geq \frac{k + 1}{\lambda_1^* + k + 2}.
\]

**Proof.** It is known that (see [3], theorem 1)

\[
\lim_{r \to \infty} \sup \frac{\log \log G(r)}{\log \log r} = \frac{\varrho_1^* + 1}{\lambda_1^* + 1}.
\]

Set \( \varrho_1^* + 1 = \mu \) and \( \lambda_1^* + 1 = \nu \). Since (3.3) is satisfied and \( 1 < \nu, \mu < \infty \), there exist a \( L \)-proximate order \( \mu(r) \) and a lower \( L \)-proximate order \( \nu(r) \) for

the function \( \log G(r) \) satisfying the conditions (i)-(v) and \((i')-(v')\) respectively in section 2 where \( F(r) \) is replaced by \( \log G(r) \).

Now, from (1.3) we have

\[
\log g_k^*(r) = \frac{k + 1}{(\log r)^{k+1}} \int_1^r (\log x) \log G(x) \frac{dx}{x} \leq
\]

\[
< O((\log r)^{k+1}) + \frac{k + 1}{(\log r)^{k+1}} \int_{r_n}^r (\log x) \log G(r) \frac{dx}{x} \sim \frac{(k + 1)(\log r)^{\mu(r)}}{(\varrho_1^* + k + 2)} (1 + O(1)),
\]

\[
r \geq r_n = \frac{(k + 1) \log G(r)}{(\varrho_1^* + k + 2)} (1 + O(1)),
\]

for \( r = r_n \to \infty \) as \( n \to \infty \). Hence, (3.1) follows.

Similarly, making use of the lower \( L \)-proximate order \( \nu(r) \) instead of \( \mu(r) \),

(3.2) is obtained.
4. - Growth of $\log G(r)$ and $\log g^*_k(r)$ relative to $u(r)$.

Theorem 4.1. Under the hypothesis of theorem 3.1, we have

$$\liminf_{r \to \infty} \frac{\log G(r)}{n(r) \log r} \leq \frac{1}{\xi_1^* + 1}, \tag{4.1}$$

$$\limsup_{r \to \infty} \frac{\log G(r)}{n(r) \log r} \geq \frac{1}{\xi_1^* + 1}, \tag{4.2}$$

$$\liminf_{r \to \infty} \frac{\log g^*_k(r)}{n(r) \log r} \leq \frac{k+1}{(\xi_1^* + 1)(\xi_1^* + k + 2)}, \tag{4.3}$$

and

$$\limsup_{r \to \infty} \frac{\log g^*_k(r)}{n(r) \log r} \geq \frac{k+1}{(\xi_1^* + 1)(\xi_1^* + k + 2)}. \tag{4.4}$$

Proof. In view of (2.1), (2.2) and $0 < \xi_1^* < \xi_1^* < \infty$, there exist a $L$-proximate order $\varrho^*_1(r)$ and a lower $L$-proximate $\lambda_1^*(r)$ relative to $n(r)$ satisfying the conditions (i)-(v) and (i')-(v)' respectively in section 2 where $F(r)$ is replace by $n(r)$.

Now, by Jensen's theorem (see Boas [1] p. 2)

$$\log G(r) = O(1) + \int_{r_0}^{r} \frac{n(x)}{x} \, dx \ll O(1) + \int_{r_0}^{r} (\log x) \varrho^*_1 \frac{dx}{x}.$$

$$\sim \frac{(\log r)\varrho_1^*_1(r) + 1}{(\xi_1^* + 1)} \frac{n(r) \log r}{\xi_1^* + 1},$$

for a sequence $r = r_n \to \infty$ as $n \to \infty$, so (4.1) is proved.

Similarly, making use of $\lambda_1^*(r)$ instead of $\varrho_1^*_1(r)$, (4.2) follows.

Further, for $1 < r_0 < r$, we have

$$\log g_1^*(r) = O((\log r)^{k+1}) + \frac{k+1}{(\log r)^{k+1}} \int_{r_0}^{r} \log G(x) \frac{dx}{x}. $$
Therefore, in view of (4.5), we see that

\[
\log g^*_x(r) \prec O((\log r)^{-k-1}) + \int_{r_0}^{r} \frac{(\log x)^{q^*_x(r)+k+1}}{q^*_x + 1} \frac{dx}{x}
\]

\[
\sim \frac{(k+1)(\log r)^{q^*_x(r)+k+1}}{(q^*_x + 1)(q^*_x + k + 2)} \left( 1 + O(1) \right)
\]

\[
= \frac{(k+1)n(r)\log r}{(q^*_x + 1)(q^*_x + k + 2)} \left( 1 + O(1) \right),
\]

for \( r = r_m \to \infty \) as \( m \to \infty \). Consequently, (4.3) is established. The inequality (4.4) can similarly be disposed of by using \( A^*_x(r) \) instead of \( q^*_x(r) \).

5. - Growth of \( \log \varphi(r) \) and \( \log g^*_x(r) \) relative to an auxiliary function involving L-proximate order.

Let \( f(x) \) be an entire function having L-order \( \varphi^* (\varphi^* < \infty) \) and L-proximate order \( \varphi^*(r) \). Further, let

\[
\varphi(r) = \frac{k+1}{(\log r)^{k+1}} \int_{r_0}^{r} (\log x)^{k} A(x) \varphi^*(x) \frac{dx}{x}, \quad A(x) = \log \varphi(x).
\]

Then \( \varphi(r) \sim \varphi^* \log g^*_x(r) \) as \( r \to \infty \). Define:

\[
\lim_{r \to \infty} \sup_{\varphi^*(r)} = \alpha, \quad \lim_{r \to \infty} \inf_{\varphi^*(r)} = \beta, \quad \lim_{r \to \infty} \sup_{\varphi^*(r)} = \gamma, \quad \lim_{r \to \infty} \inf_{\varphi^*(r)} = \delta,
\]

where

\[
\psi(r) = \exp \int_{r_0}^{r} \frac{\varphi^*(x)}{x \log x} dx, \quad \gamma > 1.
\]

Now, we prove:

Theorem 5.1:

(5.1) \[
\alpha \leq \frac{(k+1)\varphi^*\gamma}{(\varphi^* + k + 1)},
\]

(5.2) \[
\beta \leq \varphi^* \delta \left( \frac{\delta}{\gamma} \right)^{(k+1)\varphi^*} \left[ \frac{\gamma}{\delta} \left( \frac{(k+1)\varphi^*}{\varphi^* + k + 1} \right) + \frac{\varphi^*}{\varphi^* + k + 1} \right],
\]

(5.3) \[
\alpha > \frac{(k+1)\varphi^*\gamma}{(\varphi^* + k + 1)} \left[ \frac{\varphi^*\gamma}{\gamma(\varphi^* + k + 1) - \delta(k+1)} \right]^{\varphi^*(k+1)}.
\]
and

\begin{equation}
\beta \geq \frac{(k+1)\varrho^\delta}{(\varrho^+ k + 1)}.
\end{equation}

To prove this theorem, the following intermediate lemma is required:

**Lemma.** For \(0 < \eta < \infty\):

(i) \(\int_{s_0}^{r} (\log x)^{k+1+\eta} \psi'(x) \, dx \sim \frac{\varrho^+}{(\varrho^+ k + 1)} (\log r)^{k+1} \psi(r),\)

(ii) \(\int_{s_0}^{r} (\log x)^{k} g^\eta(x) \, dx \sim \frac{\varrho^+}{k+1} (\log r)^{k+1} ((1+\eta)^k - 1),\)

and

(iii) \(\frac{\psi(r^{k+1})}{\psi(r)} \sim (1+\eta)^{\varrho^+},\)

as \(r \to \infty.\)

**Proof.** We have

\[
\frac{d}{dr} ((\log r)^{k+1} \psi(r)) = (\log r)^{k+1} \psi'(r) \left(1 + \frac{(k+1)}{r \log r} \frac{\psi(r)}{\psi'(r)}\right) =
\]

\[
= (\log r)^{k+1} \psi'(r) \left(1 + \frac{k+1}{\varrho^+} \right) \sim (\log r)^{k+1} \psi(r) \left(\frac{\varrho^+}{\varrho^+} + \frac{k+1}{\varrho^+}\right),
\]

as \(r \to \infty\) and so (i) follows.

Since \(\lim_{x \to \infty} \varrho^+(x) = \varrho^+\), one can easily see that

\[
\int_{s_0}^{\infty} (\log x)^{k} g^\eta(x) \, \frac{dx}{x} \sim \varrho^+ \int_{s_0}^{\infty} (\log x)^{k} \, \frac{dx}{x},
\]

and

\[
\log \left(\frac{\psi(r^{k+1})}{\psi(r)}\right) = \int_{s_0}^{\infty} \frac{g^\eta(x)}{x \log x} \, dx \sim \varrho^+ \log (1+\eta),
\]

as \(r \to \infty\) Hence (ii) and (iii) are established.
Proof of the Theorem. Let \( 0 < \eta < \infty \) and \( r_\eta > 1 \). Then

\[
\varphi(r^{1+\eta}) = O((\log r)^{-k-1}) + \frac{(k + 1)}{(1 + \eta)^{k+1}(\log r)^{k+1}} \int_{r_\eta}^{r} (\log x)^k A(x) \varphi^\sigma(x) \frac{dx}{x}
\]

\[
+ \frac{(k + 1)}{(1 + \eta)^{k+1}(\log r)^{k+1}} \int_{r_\eta}^{r} (\log x)^k A(x) \varphi^\sigma(x) \frac{dx}{x}
\]

\[
= O((\log r)^{-k-1}) + \frac{(k + 1)}{(1 + \eta)^{k+1}(\log r)^{k+1}} \int_{r_\eta}^{r} (\log x)^k A(x) \frac{\psi'(x)}{\psi(x)} \frac{dx}{x}
\]

\[
+ \frac{(k + 1)}{(1 + \eta)^{k+1}(\log r)^{k+1}} \int_{r_\eta}^{r} (\log x)^k A(x) \varphi^\sigma(x) \frac{dx}{x}
\]

\[
< O((\log r)^{-k-1}) + \frac{(k + 1)(\gamma + \epsilon)}{(1 + \eta)^{k+1}(\log r)^{k+1}} \int_{r_\eta}^{r} (\log x)^k A(x) \frac{\psi'(x)}{\psi(x)} \frac{dx}{x}
\]

\[
= \frac{(k + 1)(\gamma + \epsilon)}{(1 + \eta)^{k+1}(\log r)^{k+1}} \int_{r_\eta}^{r} (\log x)^k \varphi^\sigma(x) \frac{dx}{x}
\]

\[
\sim \frac{(k + 1)(\gamma + \epsilon)}{(\varphi^\sigma + k + 1)(1 + \eta)^{k+1}} \frac{\varphi(\tau)}{\psi(\tau)} + \varphi^\sigma \left(1 - \frac{1}{(1 + \eta)^{k+1}}\right) A(\tau^{1+\eta}),
\]

using (i) and (ii) of the lemma. Therefore

\[
\varphi(r^{1+\eta}) < \frac{(k + 1)(\gamma + \epsilon)}{(\varphi^\sigma + k + 1)(1 + \eta)^{k+1}} \frac{\varphi(\tau)}{\psi(\tau^{2+\eta})} + \varphi^\sigma \left(1 - \frac{1}{(1 + \eta)^{k+1}}\right) A(\tau^{1+\eta}) \frac{\psi(\tau)}{\varphi(\tau^{2+\eta})}.
\]

Hence

\[
\alpha < \frac{(k + 1)\varphi^\sigma \gamma}{(\varphi^\sigma + k + 1)(1 + \eta)^{k+1} + \left(1 - \frac{1}{(1 + \eta)^{k+1}}\right) \varphi^\sigma \gamma},
\]
and
\[
\beta < \frac{(k + 1) \varrho^s \gamma}{(s^a + k + 1)(1 + \eta)^{k+1}} + \left(1 - \frac{1}{(1 + \eta)^{k+1}}\right) \varrho^s \delta.
\]

Substituting the best values of \( \eta \) namely \( \eta = 0 \) and \( \eta = (\gamma/\delta) \varrho^a - 1 \) in the above two inequalities, we get (5.1) and (5.2) respectively.

Similarly, we have
\[
\frac{\psi(r^2 + \eta)}{\psi(r^2 + \varrho^a)} > \frac{(k + 1)(\delta - \varepsilon) \varrho^s}{(s^a + k + 1)(1 + \eta)^{k+1}} + \varrho^s \left(1 - \frac{1}{(1 + \eta)^{k+1}}\right) \frac{\Lambda(r)}{\psi(r)} \frac{\psi(\varrho^s)}{\psi(r^{1+\eta})}.
\]

Therefore
\[
\alpha > \frac{(k + 1) \varrho^s \delta}{(s^a + k + 1)(1 + \eta)^{k+1} + \left(1 - \frac{1}{(1 + \eta)^{k+1}}\right) \varrho^s \gamma},
\]
and
\[
\beta > \frac{(k + 1) \varrho^s \delta}{(s^a + k + 1)(1 + \eta)^{k+1} + \left(1 - \frac{1}{(1 + \eta)^{k+1}}\right) \varrho^s \delta}.
\]

Substituting
\[
\eta = \left(\frac{(s^a + k + 1) \gamma - (k + 1) \delta}{\varrho^s \gamma}\right)^{1/(k+1)} - 1
\]
and \( \eta = 0 \) in the above inequalities, we obtain (5.3) and (5.4) respectively.

Corollary. If \( \gamma = \delta \) then it follows that
\[
\alpha = \beta = \frac{(k + 1) \varrho^s \gamma}{(s^a + k + 1)}.
\]

The converse of the result (5.5) also holds good and we prove it in the following theorem.

Theorem 5.2. If \( \alpha, \beta \) (\( 0 < \beta, \alpha < \infty \)) and \( \gamma, \delta \) (\( 0 < \delta, \gamma < \infty \)) be defined as above and if \( \alpha = \beta \), then
\[
\gamma = \delta = \frac{(s^a + k + 1) \alpha}{(k + 1) \varrho^s}.
\]
Proof. Let $0 < \eta < \infty$. Then

\[
q^\varphi((1 + \eta)^{k+1} - 1)(1 + O(1)) A(r) \geq \frac{k + 1}{(\log r)^{k+1}} A(r) \int_{r}^{r+\eta} (\log x)^k q^\varphi(x) \frac{dx}{x} < \]

\[
\leq \frac{k + 1}{(\log r)^{k+1}} \int_{r}^{r+\eta} (\log x)^k A(x) q^\varphi(x) \frac{dx}{x} = \frac{k + 1}{(\log r)^{k+1}} \left( \int_{1}^{r+\eta} (\log x)^k A(x) q^\varphi(x) \frac{dx}{x} \right. \]

\[
= (1 + \eta)^{k+1} \varphi(r^{k+1}) - \varphi(r) .
\]

Since, for arbitrarily small $\varepsilon > 0$ and $r > r_\varepsilon$, we have

\[
\varphi - \varepsilon < \frac{\varphi(r)}{\psi(r)} < \varphi + \varepsilon ,
\]

therefore

\[
q^\varphi((1 + \eta)^{k+1} - 1)(1 + O(1)) A(r) < (\varphi + \varepsilon)(1 + \eta)^{k+1} \varphi(r^{k+1}) - (\varphi - \varepsilon) \varphi(r) \sim \]

\[
\sim \varphi(r)((1 + \eta)^{k+1} - 1)\varphi + ((1 + \eta)^{k+1} + 1)\varepsilon .
\]

Hence

\[
\limsup_{r \to \infty} \frac{A(r)}{\varphi(r)} \leq \frac{\varphi((1 + \eta)^{k+1} - 1)}{q^\varphi((1 + \eta)^{k+1} - 1)} .
\]

But $\eta$ is arbitrary and so making $\eta \to 0$, we find that

\[
(5.6) \quad \limsup_{r \to \infty} \frac{A(r)}{\varphi(r)} \leq \frac{(q^\varphi + k + 1)\varphi}{q^\varphi(k + 1)} .
\]

Similarly, by considering the inequality

\[
q^\varphi(1 - (1 - \eta)^{k+1})(1 + O(1)) A(r) > \frac{k + 1}{(\log r)^{k+1}} \int_{1-\eta}^{r} (\log x)^k A(x) q^\varphi(x) \frac{dx}{x}
\]

one can readily see that

\[
(5.7) \quad \liminf_{r \to \infty} \frac{A(r)}{\varphi(r)} \geq \frac{(q^\varphi + k + 1)\varphi}{q^\varphi(k + 1)} .
\]

Hence the theorem follows from (5.6) and (5.7).
References.


Abstract.

Analogous to the properties of a proximate order and a lower proximate order for an entire function of order $q$ ($0 < q < \infty$) and lower order $\lambda$ ($0 < \lambda < \infty$), properties of a $L$-proximate order and a lower $L$-proximate order for an entire function of order zero with $L$-order $\varrho^L$ ($\varrho^L < \infty$) and lower $L$-order $\lambda^L$ ($\lambda^L < \infty$) have been considered, and used to study the various growth relations of the geometric means $G(r)$ and $g^s_1(r)$ for an entire function of order zero.

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