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Some recurrence relations involving generalized Hermite, Legendre and ultraspherical polynomials. (**)

1. - Introduction.

Sister M. Celine Fasenmyer [1] introduced an interesting technique for obtaining recurrence relation of a certain class of hypergeometric polynomials. The object of this paper is to apply her technique in the case of some generalized Hermite, Legendre and ultraspherical polynomials which have received interest in recent years. Recently L. R. Bragg [2] has considered the sequence of polynomials $\{g_n^p(x)\}$ generated by

$$\exp[ptx - t^p] = \sum_{n=0}^{\infty} \frac{g_n^p(x)}{n!} t^n . \tag{1.1}$$

The explicit representation of $g_n^p(x)$ is

$$g_n^p(x) = \sum_{s=0}^{[n/p]} \frac{(-1)^s n!}{s!(n-ps)!} (px)^{n-ps} . \tag{1.2}$$

Another generalization of the well-known Hermite polynomials is given by K. L. More [3] in the following way:

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The class of well-known Hermite polynomials may be generalized yielding a new class of orthogonal polynomials $\{H_n^\mu(x)\}$ with the weight function $\omega(x) = \exp[-x^2]|x|^\mu$, μ being a parameter, where the orthogonality relation is

$$\int_{-\infty}^{\infty} |x|^\mu \exp[-x^2] H_m^\mu(x) H_n^\mu(x)\, dx = 0\,, \qquad m \neq n\,.$$

The explicit representations of $H_n^\mu(x)$ are (n being even)

$$H_n^\mu(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k(-n/2)_k\ (-(n+\mu-1)/2)_k}{k!} x^{n-2k} \tag{1.3}$$

and (n being odd)

$$H_n^\mu(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k(-(n-1)/2)_k(-(n+\mu)/2)_k}{k!} x^{n-2k}\,, \tag{1.4}$$

where $(a)_n = a(a+1) \dots (a+n-1)$.

M. Dutta and K. L. More [4] have recently considered a new class of generalized Legendre polynomials $\{P_{n,\mu}(x)\}$, introduced with the help of the well-known Schmidt theorem and the orthogonality relation

$$\int_{-1}^{1} |x|^\mu P_{m,\mu}(x) P_{n,\mu}(x)\, dx = 0\,, \qquad m \neq n\,.$$

The explicit representations of $P_{n,\mu}(x)$ are (n being even)

$$P_{n,\mu}(x) = \sum_{k=0}^{[n/2]} \frac{(-n/2)_k(-(n+\mu-1)/2)_k}{k!(-(2n+\mu-1)/2)_k} x^{n-2k} \tag{1.5}$$

and (n being odd)

$$P_{n,\mu}(x) = \sum_{k=0}^{[n/2]} \frac{(-(n-1)/2)_k(-(n+\mu)/2)_k}{k!(-(2n+\mu-1)/2_k)} x^{n-2k}\,. \tag{1.6}$$

A. P. Barrucand [5] has considered a generalization of the well-known ultraspherical polynomials by means of the following generating relation

$$(1-ktx+t^k)^{-\lambda} = \sum_{n=0}^{\infty} c_n(\lambda, k, x)\, t^n\,. \tag{1.7}$$

It may be of interest to point out that such type of generalization is found in the work of P. Humbert [6], who was let to consider the polynomials $\pi^{\mu}_{n,m}$ where m is an integer, defined by the expansion

$$(1.8)\qquad (1-mtx+t^m)^{-\mu}=\sum_n \pi^{\mu}_{n,m}(x)\,t^n\,.$$

Here we like to change the notation adopted by Barrucand. We shall write (1.7) in the form

$$(1.9)\qquad (1-ktx+t^k)^{-\lambda}=\sum_{n=0}^{\infty} P^{\lambda}_n(x,k)\,t^n\,.$$

The recurrence relations derived in the present paper are contained in (2.6), (2.9), (2.10), (3.5), (3.9), (3.11) to (3.14), (4.2), (4.4), (4.6), (4.7), (4.9) to (4.11), (5.2), (5.4) and (5.5).

2. - Pure and differential recurrence relations for Bragg's polynomials.

The polynomial is defined by the relation (1.2). Let $\lambda^p_n(x)=\big(g^p_n(x)\big)/n!$. Then

$$\lambda^p_n(x)=\sum_{s=0}^{\infty}\frac{(-1)^s(px)^{n-ps}}{s!(n-ps)!}\,.$$

For convenience we use the upper limit of summation as infinity. We write

$$(2.1)\qquad \lambda^p_n(x)=\sum_{s=0}^{\infty}\varepsilon(s,n)\,.$$

Sister Celine's technique is to express $x\lambda^p_{n-1}(x)$, $\lambda^p_{n-p}(x)$ etc., as series involving $\varepsilon(s,n)$ and then to find a combination of coefficients which vanishes identically. Now

$$(2.2)\qquad px\lambda^p_{n-1}(x)=\sum_{s=0}^{\infty}\frac{(-1)^s(px)^{n-ps}}{s!(n-1-ps)!}=\sum_{s=0}^{\infty}(n-ps)\,\varepsilon(s,n)\,.$$

Also

$$\lambda^p_{n-p}(x)=\sum_{s=0}^{\infty}\frac{(-1)^s(px)^{n-p-ps}}{s!(n-p-ps)!}=\sum_{s=1}^{\infty}\frac{(-1)^{s-1}(px)^{n-ps}}{(s-1)!(n-ps)!}$$

so that

$$\lambda^p_{n-p}(x) = \sum_{s=0}^{\infty} - s\varepsilon(s, n) . \tag{2.3}$$

In equations (2.1)-(2.3) the co-efficients of $\varepsilon(s, n)$ are of degree one in s. Therefore, there exist constants A and B (independent of s and x) such that

$$\lambda^p_n(x) + Apx\lambda^p_{n-1}(x) + B\lambda^p_{n-p}(x) = 0 . \tag{2.4}$$

By using equations (2.1)-(2.3) we find that (2.4) is equivalent to the relation in s

$$1 + A(n - ps) - Bs = 0 .$$

Equating constant terms $A = -1/n$ and equating co-efficients of s, $B = p/n$ so that we get the pure recurrence relation

$$n\lambda^p_n(x) - px\lambda^p_{n-1}(x) + p\lambda^p_{n-p}(x) = 0 . \tag{2.5}$$

The relation for $g^p_n(x)$ is

$$\frac{g^p_n(x)}{(n-1)!} - px\frac{g^p_{n-1}(x)}{(n-1)!} + p\frac{g^p_{n-p}(x)}{(n-p)!} = 0 . \tag{2.6}$$

Again

$$D\lambda^p_n(x) = \sum_{s=0}^{\infty} \frac{(-1)^s p(n-ps)(px)^{n-ps-1}}{s!(n-ps)!} , \qquad D \equiv \frac{d}{dx} ,$$

so that

$$xD\lambda^p_n(x) = \sum_{s=0}^{\infty} (n - ps)\,\varepsilon(s, n) .$$

Also

$$\lambda^p_{n-p+1}(x) = \sum_{s=0}^{\infty} \frac{(-1)^s(px)^{n-p+1-ps}}{s!(n-p+1-ps)!} .$$

Thus

$$D\lambda_{n-p+1}^{p}(x)=\sum_{s=0}^{\infty}\frac{(-1)^{s}p(n-p+1-ps)(px)^{n-p-ps}}{s!(n-p+1-ps)!}=\sum_{s=0}^{\infty}-ps\varepsilon(s,n)\,.$$

It follows therefore that there exists a three term differential recurrence relation of the form:

$$\lambda_{n}^{p}(x)+AxD\lambda_{n}^{p}(x)+BD\lambda_{n-p+1}^{p}(x)=0 \tag{2.7}$$

where A and B are constants. We find that (2.7) is equivalent to

$$1+A(n-ps)-Bps=0$$

where from $A=-1/n$ and $B=1/n$.

Thus we obtain the relation

$$n\lambda_{n}^{p}(x)-xD\lambda_{n}^{p}(x)+D\lambda_{n-p+1}^{p}(x)=0\,. \tag{2.8}$$

The relation for $g_{n}^{p}(x)$ is

$$\frac{g_{n}^{p}(x)}{(n-1)!}-\frac{xDg_{n}^{p}(x)}{n!}+\frac{Dg_{n-p+1}^{p}(x)}{(n-p+1)!}=0\,. \tag{2.9}$$

Again from the equations for $px\lambda_{n-1}^{p}(x)$ and $xD\lambda_{n}^{p}(x)$ we find

$$p\lambda_{n-1}^{p}(x)=D\lambda_{n}^{p}(x)\,,$$

from which we get the relation in $g_{n}^{p}(x)$ as

$$Dg_{n}^{p}(x)=png_{n-1}^{p}(x)\,. \tag{2.10}$$

3. - Pure and differential recurrence relations for More's generalized Hermite polynomial.

The function $H_{n}^{\mu}(x)$ is defined by (1.3) and (1.4). *When n is even*

$$H_{n}^{\mu}(x)=\sum_{k=0}^{\infty}\frac{(-1)^{k}(-n/2)_{k}(-(n+\mu-1)/2)_{k}}{k!}\,x^{n-2k}$$

which we write as

$$H_n^\mu(x) = \sum_{k=0}^{\infty} \varepsilon(k, n) \,. \tag{3.1}$$

Also

$$xH_{n-1}^\mu(x) = \sum_{k=0}^{\infty} \frac{n-2k}{n} \varepsilon(k, n) \tag{3.2}$$

and

$$H_{n-2}^\mu(x) = \sum_{k=0}^{\infty} \frac{-4k}{n(n+\mu-1)} \varepsilon(k, n) \,. \tag{3.3}$$

It follows, therefore, from (3.1)-(3.3) that the co-efficients of $\varepsilon(k, n)$ with a lowest common denominator have their numerators of degree at most one in k, and thus we have

$$H_n^\mu(x) + AxH_{n-1}^\mu(x) + BH_{n-2}^\mu(x) = 0 \,, \tag{3.4}$$

which is equivalent to the relation in k

$$n(n+\mu-1) + A(n-2k)(n+\mu-1) - 4Bk = 0 \,,$$

from which we easily obtain $A = -1$ and $B = (n+\mu-1)/2$. Thus when n is even the pure recurrence relation is

$$H_n^\mu(x) - xH_{n-1}^\mu(x) + \frac{n+\mu-1}{2} H_{n-2}^\mu(x) = 0 \,. \tag{3.5}$$

When n is odd

$$H_n^\mu(x) = \sum_{k=0}^{\infty} \frac{(-1)^k(-(n-1)/2)_k(-(n+\mu)/2)_k}{k!} x^{n-2k} ,$$

which we write as

$$H_n^\mu(x) = \sum_{k=0}^{\infty} \varepsilon_1(k, n) \,. \tag{3.6}$$

Also

$$xH_{n-1}^\mu(x) = \sum_{k=0}^{\infty} \frac{(n+\mu-2k)}{n+\mu} \varepsilon_1(k, n) \tag{3.7}$$

and

$$H^{\mu}_{n-2}(x) = \sum_{k=0}^{\infty} \frac{-4k}{(n-1)(n+\mu)} \varepsilon_1(k, n) \,. \tag{3.8}$$

Therefore by Sister Celine's technique from (3.6)-(3.8), in a similar manner as in the case n even we obtain the pure recurrence relation when n is odd as

$$H^{\mu}_{n}(x) - xH^{\mu}_{n-1}(x) + \frac{n-1}{2} H^{\mu}_{n-2}(x) = 0 \,. \tag{3.9}$$

Differential recurrence relations:

when n is even we have

$$\left\{\begin{aligned} xDH^{\mu}_{n}(x) &= \sum_{k=0}^{\infty} (n-2k)\, \varepsilon(k, n) \,, \\ DH^{\mu}_{n-1}(x) &= \sum_{k=0}^{\infty} \frac{-4k(n+1-2k)}{n(n+\mu-2k+1)} \varepsilon(k, n) \,, \\ DH^{\mu}_{n+1}(x) &= \sum_{k=0}^{\infty} \frac{(n+\mu+1)(n+1-2k)}{(n+\mu-2k+1)} \varepsilon(k, n) \,, \\ \frac{H^{\mu}_{n-1}(x)}{x} &= \sum_{k=0}^{\infty} \frac{-4k}{n(n+\mu-2k+1)} \varepsilon(k, n) \,. \end{aligned}\right. \tag{3.10}$$

In the equations (3.10) the maximum degree with respect to k of the numerators of the co-efficients of $\varepsilon(k, n)$ when taken with a lowest common denominator is two. Therefore, there will exist a four term differential recurrence relation connecting $H^{\mu}_{n}(x)$, $xH^{\mu}_{n-1}(x)$, $\big(H^{\mu}_{n-1}(x)/x\big)$ and $DH^{\mu}_{n-1}(x)$. Let

$$H^{\mu}_{n}(x) + AxH^{\mu}_{n-1}(x) + BDH^{\mu}_{n-1}(x) + \frac{CH^{\mu}_{n-1}(x)}{x} = 0 \,,$$

which gives

$$n(n+\mu-2k+1) + A(n-2k)(n+\mu-2k+1) - 4Bk(n+1-2k) - 4Ck = 0 \,.$$

We get $A = -1$, $B = 1/2$ and $C = \mu/2$.

Thus the differential recurrence relation is

$$(3.11) \qquad 2xH_n^\mu(x) - 2x^2 H_{n-1}^\mu(x) + xDH_{n-1}^\mu(x) + \mu H_{n-1}^\mu(x) = 0 .$$

It also follows from (3.2) and (3.10) that

$$(3.12) \qquad nH_{n-1}^\mu(x) = DH_n^\mu(x) .$$

When n is odd, we have

$$xDH_n^\mu(x) = \sum_{k=0}^{\infty} (n-2k)\,\varepsilon_1(k, n) , \qquad DH_{n-1}^\mu(x) = \sum_{k=0}^{\infty} \frac{-4k}{n+\mu}\,\varepsilon_1(k, n) .$$

Thus there exists a relation

$$H_n^\mu(x) + AxH_{n-1}^\mu(x) + BDH_{n-1}^\mu(x) = 0 ,$$

which is equivalent to

$$(n+\mu) + A(n+\mu-2k) - 4Bk = 0 ,$$

so that $A = -1$ and $B = \frac{1}{2}$. Thus we obtain the differential recurrence relation

$$(3.13) \qquad 2H_n^\mu(x) - 2xH_{n-1}^\mu(x) + DH_{n-1}^\mu(x) = 0 .$$

We may have another recurrence relation connecting $H_n^\mu(x)$, $xH_{n-1}^\mu(x)$ and $xDH_n^\mu(x)$. In a similar manner as above we obtain the differential recurrence relation

$$(3.15) \qquad \mu H_n^\mu(x) - (n+\mu)\,xH_{n-1}^\mu(x) + xDH_n^\mu(x) = 0 .$$

4. - Pure and differential recurrence relations for Dutta and More's generalized Legendre polynomials.

The polynomial is defined by (1.5) when n is even and by (1.6) when n is odd. When n is even, we have

$$P_{n,\mu}(x) = \sum_{k=0}^{\infty} \frac{(-n/2)_k(-(n+\mu-1)/2)_k}{k!(-(2n+\mu-1)/2)_k}\, x^{n-2k} ,$$

which is written as

$$P_{n,\mu}(x) = \sum_{k=0}^{\infty} \varepsilon(k, n) .$$

Thus

$$xP_{n-1,\,\mu}(x) = \sum_{k=0}^{\infty} \frac{(n-2k)(2n+\mu-1)}{n(2n+\mu-2k-1)}\, \varepsilon(k, n)$$

and

$$P_{n-2,\,\mu}(x) = \sum_{k=0}^{\infty} \frac{-2k(2n+\mu-1)(2n+\mu-3)}{n(n+\mu-1)(2n+\mu-2k-1)}\, \varepsilon(k, n) .$$

In the above equations when the co-efficients of $\varepsilon(k, n)$ have a lowest common denominator, the maximum degree with respect to k of the numerators is one. Then there exist constants A and B such that

$$P_{n,\mu}(x) + AxP_{n-1,\mu}(x) + BP_{n-2,\mu}(x) = 0 . \tag{4.1}$$

By using the above equations we find that (4.1) is equivalent to the identity in k

$$n(n+\mu-1)(2n+\mu-2k-1) + A(n-2k)(2n+\mu-1)\cdot$$
$$\cdot(n+\mu-1) - 2Bk(2n+\mu-1)(2n+\mu-3) = 0 .$$

By usual process we get

$$A = -1, \qquad B = \frac{(n+\mu-1)^2}{(2n+\mu-1)(2n+\mu-3)} .$$

Therefore, when n is even, the pure recurrence relation is

$$(2n+\mu-1)(2n+\mu-3)P_{n,\mu}(x) - (2n+\mu-1)(2n+\mu-3)\cdot$$
$$\cdot xP_{n-1,\mu}(x) + (n+\mu-1)^2 P_{n-2,\mu}(x) = 0 . \tag{4.2}$$

When n is odd, we have

$$P_{n,\mu}(x) = \sum_{k=0}^{\infty} \frac{(-(n-1)/2)_k(-(n+\mu)/2)_k}{k!(-(2n+\mu-1)/2)_k}\, x^{n-2k},$$

which we write as

$$P_{n,\mu}(x) = \sum_{k=0}^{\infty} \varepsilon_1(k, n) .$$

Thus

$$xP_{n-1,\,\mu}(x) = \sum_{k=0}^{\infty} \frac{(n+\mu-2k)(2n+\mu-1)}{(n+\mu)(2n+\mu-2k-1)} \varepsilon_1(k, n) .$$

Also

$$P_{n-2,\,\mu}(x) = \sum_{k=0}^{\infty} \frac{-2k(2n+\mu-1)(2n+\mu-3)}{(n-1)(n+\mu)(2n+\mu-2k-1)} \varepsilon_1(k, n) .$$

Therefore, there exist constants A and B such that

$$P_{n,\mu}(x) + AxP_{n-1,\mu}(x) + BP_{n-2,\mu}(x) = 0 , \tag{4.3}$$

which is equivalent to

$$(n+\mu)(n-1)(2n+\mu-2k-1) + A(n-1)(n+\mu-2k)(2n+\mu-1) - \\ - 2Bk(2n+\mu-1)(2n+\mu-3) = 0 .$$

Calculating A and B we get the pure recurrence relation when n is odd as

$$(2n+\mu-1)(2n+\mu-3)\, P_{n,\mu}(x) - \\ (2n+\mu-1)(2n+\mu-3)\, xP_{n-1,\mu}(x) + (n-1)^2 P_{n-2,\mu}(x) = 0 . \tag{4.4}$$

Differential recurrence relations:
when n is even we have

$$xDP_{n,\mu}(x) = \sum_{k=0}^{\infty} (n-2k)\, \varepsilon(k, n) ,$$

$$DP_{n-1,\mu}(x) = \sum_{k=0}^{\infty} \frac{-2k(n+1-2k)(2n+\mu-1)}{n(n+\mu-2k+1)} \varepsilon(k, n) ,$$

$$DP_{n+1,\mu}(x) = \sum_{k=0}^{\infty} \frac{(n+1-2k)(n+\mu+1)(2n+\mu-2k+1)}{(2n+\mu+1)(n+\mu-2k+1)} \varepsilon(k, n) ,$$

$$\frac{P_{n-1,\mu}(x)}{x} = \sum_{k=0}^{\infty} \frac{-2k(2n+\mu-1)}{n(n+\mu-2k+1)} \varepsilon(k, n) .$$

There exists a four term recurrence relation connecting $P_{n,\mu}(x)$, $xDP_{n,\mu}(x)$, $DP_{n-1,\mu}(x)$ and $(P_{n-1,\mu}(x))/x$. Thus constants A, B and C exist such that

$$P_{n,\mu}(x) + AxDP_{n,\mu}(x) + BDP_{n-1,\mu}(x) + \frac{CP_{n-1,\mu}(x)}{x} = 0 . \tag{4.5}$$

We find that (4.5) is equivalent to

$$n(n+\mu-2k+1) + An(n-2k)(n+\mu-2k+1) - \\ - 2Bk(n+1-2k)(2n+\mu-1) - 2Ck(2n+\mu-1) = 0 ,$$

from which $A = -1/n$, $B = 1/(2n+\mu-1)$ and $C = \mu/(2n+\mu-1)$.

Thus the recurrence relation is

$$n(2n+\mu-1)\,xP_{n,\mu}(x) - (2n+\mu-1)\,x^2DP_{n,\mu}(x) + \\ + nxDP_{n-1,\mu}(x) + n\mu P_{n-1,\mu}(x) = 0 . \tag{4.6}$$

Again there exists a four-term recurrence relation connecting $P_{n,\mu}(x)$, $DP_{n-1,\mu}(x)$, $DP_{n+1,\mu}(x)$ and $xDP_{n,\mu}(x)$. In a similar manner as above we get the differential recurrence relation

$$(2n+\mu-1)(2n+\mu+1)\,P_{n,\mu}(x) - n^2DP_{n-1,\mu}(x) - \\ - (2n+\mu-1)(2n+\mu+1)\,DP_{n+1,\mu}(x) + \\ + (2n+\mu-1)(2n+\mu+1)\,xDP_{n,\mu}(x) = 0 . \tag{4.7}$$

When n is odd, we have

$$xDP_{n,\mu}(x) = \sum_{k=0}^{\infty} (n-2k)\,\varepsilon_1(k,n),$$

$$DP_{n-1,\mu}(x) = \sum_{k=0}^{\infty} \frac{-2k(2n+\mu-1)}{n+\mu}\,\varepsilon_1(k,n) ,$$

$$DP_{n+1,\mu}(x) = \sum_{k=0}^{\infty} \frac{(n+1)(2n+\mu-2k+1)}{2n+\mu+1}\,\varepsilon_1(k,n) .$$

The above equations suggest that there exists a relation

$$P_{n,\mu}(x) + AxDP_{n,\mu}(x) + BDP_{n-1,\mu}(x) = 0 , \tag{4.8}$$

A and B being constants. Now (4.8) is equivalent to

$$(n+\mu)+A(n+\mu)(n-2k)-2Bk(2n+\mu-1)=0\,,$$

from which $A=-1/n$ and $B=(n+\mu)/\big(n(2n+\mu-1)\big)$. Thus we have the relation

$$\begin{aligned}(4.9)\qquad & n(2n+\mu-1)\,P_{n,\mu}(x)-(2n+\mu-1)\,xDP_{n,\mu}(x)+\\ &\qquad +(n+\mu)\,DP_{n-1,\mu}(x)=0\,.\end{aligned}$$

Similarly we obtain the relations

$$\begin{aligned}(4.10)\qquad & (n+1)(2n+\mu-1)(2n+\mu+1)\,P_{n,\mu}(x)+\\ & +(n+1)(n+\mu)DP_{n-1,\mu}(x)-(2n+\mu-1)(2n+\mu+1)\,DP_{n+1,\mu}(x)=0\end{aligned}$$

and

$$\begin{aligned}(4.11)\qquad & (n+1)(n+\mu+1)\,P_{n,\mu}(x)+(n+1)\,xDP_{n,\mu}(x)-\\ &\qquad -(2n+\mu+1)\,DP_{n+1,\mu}(x)=0\,.\end{aligned}$$

5. - Pure and differential recurrence relations for generalized ultraspherical polynomials.

From (1.9) we have

$$P_n^\lambda(x,k)=\sum_{p=0}^{[n/k]}\frac{(-1)^p(\lambda)_{n-(k-1)p}\,(kx)^{n-kp}}{p\,!(n-kp)\,!}\,,$$

which we write as

$$P_n^\lambda(x,k)=\sum_{p=0}^{\infty}\frac{(-1)^p(\lambda)_{n-(k-1)p}(kx)^{n-kp}}{p\,!(n-kp)\,!}=\sum_{p=0}^{\infty}\varepsilon(p,n)\,.$$

Thus

$$xP_{n-1}^\lambda(x,k)=\sum_{p=0}^{\infty}\frac{(n-kp)}{k\{\lambda+n-(k-1)p-1\}}\;\varepsilon(p,n)$$

and

$$P_{n-k}^\lambda(x,k)=\sum_{p=0}^{\infty}\frac{-p}{\{\lambda+n-(k-1)\,p-1\}}\;\varepsilon(p,n)\,.$$

In $P_n^\lambda(x,k)$, $xP_{n-1}^\lambda(x,k)$ and $P_{n-k}^\lambda(x,k)$ coefficients of $\varepsilon(p,n)$ have their numerators of the first degree in p when a common denominator is used. Therefore, there exists a recurrence relation of the form

$$P_n^\lambda(x,k)+AxP_{n-1}^\lambda(x,k)+BP_{n-k}^\lambda(x,k)=0 \tag{5.1}$$

which is equivalent to

$$\{\lambda+n-(k-1)p-1\}k+A(n-kp)-Bpk=0$$

and from this $A=\big(k(1-n-\lambda)\big)/n$ and $B=\big(k(\lambda-1)+n\big)/n$.

Thus we get the pure recurrence relation

$$nP_n^\lambda(x,k)+(1-n-\lambda)kxP_{n-1}^\lambda(x,k)+\{k(\lambda-1)+n\}P_{n-k}^\lambda(x,k)=0 \tag{5.2}$$

which was obtained by A. P. Barrucand in a different method.

Differential recurrence relations:

$$\begin{aligned}
DP_{n+1}^\lambda(x,k) &= \sum_{p=0}^{\infty} k\{\lambda+n-(k-1)p\}\,\varepsilon(p,n)\,,\\
xDP_n^\lambda(x,k) &= \sum_{p=0}^{\infty}(n-kp)\,\varepsilon(p,n)\,,\\
DP_{n-k+1}^\lambda(x,k) &= \sum_{p=0}^{\infty}-kp\varepsilon(p,n)\,.
\end{aligned}$$

There will exist a three term differential recurrence relation of the type

$$P_n^\lambda(x,k)+ADP_{n+1}^\lambda(x,k)+BDP_{n-k+1}^\lambda(x,k)=0 \tag{5.3}$$

which is equivalent to

$$1+Ak\{\lambda+n-(k-1)p\}-Bpk=0$$

giving $A=-1/\big(k(\lambda+n)\big)$ and $B=(k-1)/\big(k(\lambda+n)\big)$ so that the differential recurrence relation is

$$k(\lambda+n)P_n^\lambda(x,k)-DP_{n+1}^\lambda(x,k)+(k-1)DP_{n-k+1}^\lambda(x,k)=0\,. \tag{5.4}$$

Again, from the equations for $P_n^\lambda(x,k)$, $xDP_n^\lambda(x,k)$ and $DP_{n-k+1}^\lambda(x,k)$, we easily get the differential recurrence relation

$$nP_n^\lambda(x, k) - xDP_n^\lambda(x, k) + DP_{n-k+1}^\lambda(x, k) = 0 \tag{5.5}$$

which was derived by S. K. Chatterjee and B. B. Saha [7] in a different way.

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