Riv. Mat. Univ. Parma (4) 1 (1976), 21-32

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Certain theorems on generalised Hankel transform. (**)

1. - Introduction.

Let

(1.1) $$\psi(p) = p \int_0^\infty \exp[-pt] f(t)\, dt\,, \qquad R(p) > 0\,,$$

then we say that $\psi(p)$ *is operationally related to* $f(t)$ and simbolically we write

(1.2) $$\psi(p) \doteqdot f(t) \qquad \text{or} \qquad f(t) \doteqdot \psi(p)\,.$$

If $f(t) \doteqdot \psi(p)$ then McLachlan and Humbert [5] proved the following results:

(1.3) $$p^{1-v}\psi(1/p) \doteqdot t^{v/2} \int_0^\infty x^{-v/2} J_v(2\sqrt{xt}) f(x)\, dx\,,$$

(1.4) $$t^v f(1/t) \doteqdot p^{1/2-v/2} \int_0^\infty x^{v/2-1/2} J_{v+1}(2\sqrt{xp})\, \psi(x)\, dx\,.$$

We shall use these results in our discussion.

Also Goldstein [2] has proved that if $f(t) \doteqdot \psi(p)$ and $g(t) \doteqdot \varphi(p)$, then

(1.5) $$\int_0^\infty \varphi(t) f(t)\, dt/t = \int_0^\infty \psi(t) g(t)\, dt/t$$

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provided the necessary changes in the order of integrations are permissible and the integrals converge.

Also « if $f(t)$ is a continuous function satisfying (1.1), then it is the only continuous function doing so ». This theorem is due to Lerch [3].

In the present Note we shall investigate that if $\psi(p)$ and $\varphi(p)$ are related to each other in some way, then $f(t)$ and $g(t)$ will also be related to each other in a similar manner, $f(t)$ and $g(t)$ being operationally represented by $\psi(p)$ and $\varphi(p)$ respectively.

In 1931, G. N. Watson [8] proved that the function

$$\tilde{\omega}_{u,v}(x) = \sqrt{x} \int_0^\infty J_u(xt)\, J_v(1/t)\, dt/t \tag{1.6}$$

$R(u, v) \geqslant -\frac{1}{2}$, is a Fourier kernel.

In 1953, Bhatnagar [1] generalised this kernel and the generalise kernel is defined as

$$\tilde{\omega}_{u_1,u_2,\dots u_n}(x) = \int_0^\infty \tilde{\omega}_{u_1,u_2,\dots,u_{n-1}}(x/y)\, J_u(y)\, dy/\sqrt{y}\,, \tag{1.7}$$

$R(u_1, u_2, \dots, u_n) \geqslant -\frac{1}{2}$, where $u_1, u_2, \dots, u_n$ can be permuted among themselves without altering the function. Also

$$\tilde{\omega}_{u_1,\dots,u_n}(x) = O(x^{u_1+1/2}, \dots, x^{u_n+1/2}) \qquad \text{for small } x\,; \tag{1.8}$$

$$\tilde{\omega}_{u_1\dots,u_n}(x) = O[x^{-(n-1)/2n}] \qquad \text{for large } x\,. \tag{1.9}$$

Two functions $f(x)$ and $g(x)$ are called $\tilde{\omega}_{u_1,u_2\dots,u_n}(x)$ *transform of each other* if they satisfy the integral equation

$$f(x) = \int_0^\infty \tilde{\omega}_{u_1,\dots,u_n}(xy)\, g(y)\, dy\,. \tag{1.10}$$

If $g(x) = f(x)$, i.e., $f(x) = \int_0^\infty \tilde{\omega}_{u_1,\dots,u_n}(xy) f(y)\, dy$, then $f(x)$ is said to be *self-reciprocal in* $\tilde{\omega}_{u_1,\dots,u_n}(x)$ and is denoted by $R_{u_1,u_2,\dots,u_n}$.

In 1963, V. P. Mainra [4] has defined the kernel

$$\tilde{\omega}^{v_1,\dots,v_m}_{u_1,\dots,u_n}(x) = \int_0^\infty \tilde{\omega}^{v_1,\dots,v_{m-1}}_{u_1,\dots,u_n}(xy)\, J_{v_m}(y)\, \sqrt{y}\, dy\,, \tag{1.11}$$

$m < n\,,\quad R(u_r, v_s) \geqslant -\frac{1}{2}$ for all $r = 1, 2, \dots, n;\ s = 1, 2, \dots, m.$

Two functions $f(x)$ and $g(x)$ are called $\tilde{w}^{v_1,\dots,v_m}_{u_1,\dots,u_n}(x)$ *transforms of each other* if they satisfy the integral equation

$$f(x) = \int_0^\infty \tilde{w}^{v_1,\dots,v_m}_{u_1,\dots,u_n}(xy)\, g(y)\, dy \tag{1.12}$$

and the function which is self-reciprocal under this kernel is denoted by $R^{v_1,\dots,v_m}_{u_1,\dots,u_n}$; when $u_1 = v_1,\, u_2 = v_2,\, \dots,\, u_m = v_m$, then (1.11) reduces to $\tilde{w}_{u_{m+1},u_{m+2},\dots,u_n}(x)$.

In particular, $\tilde{w}^{\lambda}_{u,v}(x) = \int_0^\infty \tilde{w}_{u,v}(xt) J_\lambda(t)\sqrt{t}\, dt$, is the resultant of $\sqrt{x}J_\lambda(x)$ and $\tilde{w}_{uv}(x)$, when $\lambda = v$, this reduces to $\sqrt{x}J_u(x)$.

$$\tilde{w}^{v_1,\dots,v_m}_{u_1,\dots,u_n}(x) = O(x^{u_r+1/2}) \qquad (r = 1, 2, \dots, n) \qquad \text{for small } x\,, \tag{1.13}$$

$$\tilde{w}^{v_1\dots v_m}_{u_1\dots u_n}(x) = O(x^{-(n-m-1)/2(n-m)}) \qquad \text{for large } x\,. \tag{1.14}$$

Notations. In this paper we shall denote the functions

$$\tilde{w}^{v_1,\dots,v_m}_{u_1,\dots,u_n}(x)\,, \qquad \tilde{w}^{v_1,v_2,\dots,v_m}_{u_1,u_2,\dots,u_n,v}(x)\,, \qquad \tilde{w}^{v_1,\dots,v_m,v}_{u_1,\dots,u_n}(x)$$

by $\overline{W}^m_n(x)$, $\overline{W}^m_{n,v}(x)$, $\overline{W}^{m,v}_n(x)$ respectively. Also we shall denote the function which is self-reciprocal, under these kernels by $\overline{R}^m_n(x)$, $\overline{R}^m_{n,v}(x)$, $\overline{R}^{m,v}_n(x)$ respectively.

2. – Let

$$f(t) \doteqdot \psi(p) \tag{2.1}$$

and

$$F(at) \doteqdot \varphi(p/a)\,. \tag{2.2}$$

Applying Goldstein's theorem to (2.1) and (2.2) we get

$$\int_0^\infty f(x)\,\varphi(x/a)\, dx/x = \int_0^\infty \psi(x) F(ax)\, dx/x\,. \tag{2.3}$$

By putting $a = 1/p$ and interpreting with the help of (2.2) we get

$$\int_0^\infty F(y/x) f(x)\, dx/x \doteqdot \int_0^\infty F(x/p)\,\psi(x)\, dx/x\,. \tag{2.4}$$

Let us put $F(t) = t^k \overline{W}_n^m(1/t)$ in (2.4), we get

$$y^k \int_0^\infty \overline{W}_n^m(x/y)\, x^{-k-1} f(x)\, dx \doteqdot p^{-k} \int_0^\infty \overline{W}_n^m(p/x)\, x^{k-1} \psi(x)\, dx\,, \tag{2.5}$$

$$R\left(k + \frac{n-m-1}{2n-2m}\right) > -1\,.$$

Let

$$f_1(y) = y^k \int_0^\infty \overline{W}_n^m(x/y)\, x^{-k-1} f(x)\, dx \tag{2.6}$$

and

$$\psi_1(p) = p^{-k} \int_0^\infty \overline{W}_n^m(p/x)\, x^{k-1} \psi(x)\, dx\,. \tag{2.7}$$

Hence we get $f_1(y) \doteqdot \psi_1(p)$.

MITRA and BOSE [**6**] proved that if $f_1(y) \doteqdot \psi_1(p)$, then

$$t^{v+1} \int_0^\infty J_v(tz)\, t^{-v} f_1(z)\, dz \doteqdot p^{1-v} \int_0^\infty J_{v+1}(pz)\, z^v \psi_1(z)\, dz\,, \tag{2.8}$$

provided $z^{-v} f_1(z)$ and $z^{v-1/2}\psi_1(z)$ are continuous and absolutely integrable in $(0, \infty)$ and $R(v) > -1$.

By substituting the values of $f_1(z)$ and $\psi_1(z)$ from (2.6) and (2.7) in (2.8), we have

$$t^{v+1} \int_0^\infty z^{k-v} J_v(tz)\, dz \int_0^\infty \overline{W}_n^m(x/z)\, x^{-k-1} f(x)\, dx \doteqdot$$
$$\doteqdot p^{1-v} \int_0^\infty z^{v-k} J_{v+1}(pz)\, dz \int_0^\infty \overline{W}_n^m(z/x)\, x^{k-1}\, dx\,.$$

By changing the order of integrations on both sides and putting $k = v - \frac{1}{2}$, we have

$$t^{v+1} \int_0^\infty x^{-v-1/2} f(x)\, dx \int_0^\infty \overline{W}_n^m(x/z)\, J_v(tz)\, dz/\sqrt{z} \doteqdot$$
$$\doteqdot p^{1-v} \int_0^\infty x^{v-3/2} \psi(x)\, dx \int_0^\infty \overline{W}_n^m(z/x)\, J_{v+1}(pz)\sqrt{z}\, dz,$$

provided $x^{v-1/2}\psi(x)$ and $x^{-v}f(x)$ are continuous and absolutely integrable in $(0, \infty)$ and $R(u_1, \dots, u_n, v_1, \dots, v_m) \geqslant -\frac{1}{2}$, $n > m+2$.

On putting z/t for z on the left hand side and z/p for z on the right hand side we have

$$(2.9)\qquad t^{v+1/2} \int_0^\infty \overline{W}^m_{n,v}(xt)\, x^{-v-1/2} f(x)\, \mathrm{d}x \doteqdot p^{-v-1/2} \int_0^\infty \overline{W}^{m,\, v+1}_n(x/p)\, x^{-v-1/2}\, \psi(1/x)\, \mathrm{d}x\,.$$

Let us put

$$\varphi(p) = p^{-v-1/2} \int_0^\infty \overline{W}^{m,\, v+1}_n(x/p)\, x^{-v-1/2}\, \psi(1/x)\, \mathrm{d}x\,,$$

i.e. $y^{-v-1/2}\varphi(1/y)$ is the $\overline{W}^{m,\, v+1}_n(y)$ transform of $x^{-v-1/2}\psi(1/x)$.

If $g(t) \doteqdot \varphi(p)$, then we have from (2.9)

$$y^{-v-1/2} g(y) = \int_0^\infty x^{-v-1/2} f(x)\, \overline{W}^m_{n,v}(xt)\, \mathrm{d}x$$

(provided both sides are continuous functions of y) which shows that $y^{-v-1/2}g(y)$ is the $\overline{W}^m_{n,v}(x)$ transform of $x^{-v-1/2}f(x)$.

Hence we state the following theorems:

Theorem 1(a). *Let $f(t) \doteqdot \psi(p)$, $g(t) \doteqdot \varphi(p)$ and $x^{-v-1/2}\varphi(1/x)$ be the $\overline{W}^{m,\, v+1}_m$ transform of $y^{-v-1/2}\psi(1/y)$. Then $x^{-v-1/2}g(x)$ will be the $\overline{W}^m_{n,v}(x)$ transform of $y^{-v-1/2}f(y)$, provided that $x^{v-1/2}\psi(x)$ and $x^{-v}f(x)$ are continuous and are absolutely integrable in $(0, \infty)$. Also $g(t)$ and $t^{v+1/2}\int_0^\infty x^{-v-1/2} f(x)\, \overline{W}^m_{n,v}(xt)\, \mathrm{d}x$ are continuous functions of t and $R(u_1, \dots, u_n, v_1, \dots, v_m) \geqslant -\frac{1}{2}$.*

Theorem 1(b). *Let $f(t) \doteqdot \psi(p)$, $g(t) \doteqdot \varphi(p)$ and $x^{-v-1/2}g(x)$ be the $\overline{W}^m_{n,v}(x)$ transform of $y^{-v-1/2}f(y)$. Then $x^{-v-1/2}\varphi(1/x)$ will be the $\overline{W}^{m,\, v+1}_n(x)$ transform of $y^{-v-1/2}\psi(1/y)$, provided the conditions of the Theorem 1(a) are satisfied.*

If we take $n = 3$, $m = 0$, $u_1 = u$, $u_2 = \lambda$ and $u_3 = v+1$, the Theorem 1(a) reduces to the following

Corollary 1. *Let $f(t) \doteqdot \psi(p)$, $g(t) \doteqdot \varphi(p)$ and let $x^{-v-1/2}\varphi(1/x)$ the $\tilde{w}_{u,\lambda}(x)$ transform of $y^{-v-1/2}\psi(1/y)$. Then $x^{-v-1/2}g(x)$ will the $\tilde{w}_{u,\lambda\, v+1,v}(x)$ transform of $y^{-v-1/2}f(y)$, provided that the conditions of the theorem are satisfied.*

Further let $x^{-v-1/2}\psi(1/x)$ be $\overline{R}^{m,\, v+1}_n$, then from (2.9) we have

$$(2.10)\qquad t^{v+1/2} \int_0^\infty x^{-v-1/2} f(x)\, \overline{W}^m_{n,v}(xt)\, \mathrm{d}x \doteqdot \psi(p)\,.$$

Since $f(t) \doteqdot \psi(p)$ then by Lerch's theorem we have from (2.10)

$$t^{-v-1/2} f(t) = \int_0^\infty x^{-v-1/2} f(x) \, \overline{W}^m_{n,v}(xt) \, dx \,, \tag{2.11}$$

provided $f(t)$ and $t^{v+1/2}\int_0^\infty x^{-v-1/2} f(x) \, \overline{W}^m_{n,v}(xt) \, dx$ are continuous functions of t. Hence from (2.11) we get $x^{-v-1/2} f(x)$ is $\bar{R}^m_{n,v}$. Conversely, if $x^{-v-1/2} f(x)$ is $\bar{R}^m_{n,v}$ in (2.9) then we can prove that $x^{-v-1/2}\psi(1/x)$ will be $\bar{R}^{m,\,v+1}_n$. Thus we state the following corollaries.

Corollary 2. *Let $f(t) \doteqdot \psi(p)$ and let $x^{-v-1/2}\psi(1/x)$ be $\bar{R}^{m,\,v+1}_n$. Then $x^{-v-1/2} f(x)$ will be $\bar{R}^m_{n,v}$, provided $x^{-v-1/2} f(x)$ be continuous and absolutely integrable in $(0, \infty)$.*

Corollary 3. *Let $f(t) \doteqdot \psi(p)$ and let $x^{-v-1/2} f(x)$ be $\bar{R}^m_{n,v}$. Then $x^{-v-1/2}\psi(1/x)$ will be $\bar{R}^{m,\,v+1}_n$, provided $x^{-v-1/2}\psi(1/x)$ is continuous and absolutely integrable in $(0, \infty)$.*

In particular, when $m = 0$, $n = 3$, $u_1 = u$, $u_2 = \lambda$ and $u_3 = v + 1$ then we have from the Corollary 2.

Corollary 4. *Let $f(t) \doteqdot \psi(p)$ and let $x^{-v-1/2}\psi(1/x)$ be $R_{u,\lambda}$, then $x^{-v-1/2} f(x)$ will be $R_{u,\lambda,\,v+1,v}$, provided the conditions of the Corollary 2 are satisfied.*

Let us put $F(t) = t^k \, \overline{W}^m_n(\sqrt{t})$ in (2.4); we get

$$y^k \int_0^\infty \overline{W}^m_n(\sqrt{y/x}) \, x^{-k-1} f(x) \, dx \doteqdot p^{-k} \int_0^\infty x^{k-1} \psi(x) \, \overline{W}^m_n(\sqrt{x/p}) \, dx \,, \tag{2.12}$$

provided $x^{-k} f(x)$ and $x^{k-1}\psi(x)$ are continuous and absolutely integrable in $(0, \infty)$.

On writing x^2 for x on both sides of (2.12), we get

$$y^k \int_0^\infty x^{-2k-1} f(x^2) \, \overline{W}^m_n(\sqrt{y}/x) \, dx \doteqdot p^{-k} \int_0^\infty x^{2k-1} \psi(x^2) \, \overline{W}^m_n(x/\sqrt{p}) \, dx \,. \tag{2.13}$$

Let us put

$$g_1(y) = y^k \int_0^\infty x^{-2k-1} f(x^2) \, \overline{W}^m_n(\sqrt{y}/x) \, dx \tag{2.14}$$

and

$$\varphi_1(p) = p^{-k} \int_0^\infty x^{2k-1} \psi(x^2) \, \overline{W}^m_n(x/\sqrt{p}) \, dx \,. \tag{2.15}$$

Hence we get $g_1(y) \doteqdot \varphi_1(p)$, then from (1.4) we have

$$y^v g_1(1/y) \doteqdot p^{1/2-v/2} \int_0^\infty x^{v/2-1/2} J_{v+1}(2\sqrt{px})\, \varphi_1(x)\, dx . \tag{2.16}$$

On substituting the values of $g_1(1/y)$ and $\varphi_1(x)$ from (2.14) and (2.15) in (2.16), we have

$$y^{v-k} \int_0^\infty x^{-2k-1} f(x^2)\, \overline{W}{}_n^m(1/x\sqrt{y})\, dx \doteqdot$$
$$\doteqdot p^{1/2-v/2} \int_0^\infty t^{v/2-k-1/2} \cdot J_{v+1}(2\sqrt{pt})\, dt \int_0^\infty x^{2k-1} \psi(x^2)\, \overline{W}{}_n^m(x/\sqrt{t})\, dx ,$$

taking $k = v/2 + \frac{1}{4}$ and changing the order of integrations on the right hand side we have

$$y^{v/2-1/2} \int_0^\infty x^{-v-3/2} f(x^2)\, \overline{W}{}_n^m(1/x\sqrt{y})\, dx \doteqdot$$
$$\doteqdot p^{1/2-v/2} \int_0^\infty x^{v-1/2} \psi(x^2)\, dx \cdot \int_0^\infty \overline{W}{}_n^m(x/\sqrt{t}) J_{v+1}(2\sqrt{pt})\, t^{-3/4}\, dt,$$

$R(u_r + \frac{1}{2}) \geqslant 0,\ r = 1, 2, \ldots, n,\ R(v_s + \frac{1}{2}) \geqslant 0,\ s = 1, 2, \ldots, m.$

On writing $t^2/4p$ for t on the right hand side and $1/x$ for x on the left hand side, we have

$$y^{v/2-1/4} \int_0^\infty x^{v-1/2} f(1/x^2)\, \overline{W}{}_n^m(x/\sqrt{y})\, dx \doteqdot$$
$$\doteqdot \sqrt{2}\, p^{1/4-v/2} \cdot \int_0^\infty x^{v-1/2} \psi(x^2)\, \overline{W}{}_{n,\,v+1}^m(2x\sqrt{p})\, dx . \tag{2.17}$$

Let $y^{v-1/2} g(1/y^2)$ be the $\overline{W}{}_n^m(x)$ transform of $t^{v-1/2} f(1/t^2)$ and let $g(t) \doteqdot \varphi(p)$, we have from (2.17)

$$p^{v-1/2} \varphi(p^2/2) = \int_0^\infty x^{v-1/2} \psi(x^2/2)\, \overline{W}{}_{n,v+1}^m(px)\, dx ,$$

which shows that $p^{v-1/2}\varphi(p^2/2)$ is the $\overline{W}{}_{n,\,v+1}^m(x)$ transform of $x^{v-1/2}\psi(x^2/2)$. Hence we state the theorems as.

Theorem 2(a). *Let $f(t) \doteqdot \psi(p), g(t) \doteqdot \varphi(p)$ and let $x^{v-1/2}g(1/x^2)$ be the $\overline{W}{}_n^m(x)$ transform of $y^{v-1/2}f(1/y^2)$. Then $x^{v-1/2}\varphi(x^2/2)$ will be the $\overline{W}{}_{n,v+1}^m(x)$ transform of $y^{v-1/2}\psi(y^2/2)$, provided $y^{v-1/2}\psi(y^2/2)$ is continuous and absolutely integrable in* $(0, \infty)$.

Conversely we state the theorem as following

Theorem 2(b). *Let* $f(t) \doteqdot \psi(p)$, $g(t) \doteqdot \varphi(p)$ *and let* $x^{v-1/2}\varphi(x^2/2)$ *be the* $\overline{W}^m_{n,v+1}(x)$ *transform of* $y^{v-1/2}\psi(y^2/2)$, *then* $x^{v-1/2}g(1/x^2)$ *will be* $\overline{W}^m_n(x)$ *transform of* $y^{v-1/2}f(1/y^2)$, *provided the conditions of the Theorem* 2(a) *are satisfied.*

Under the conditions of the Theorem 2(a) we have the following corollaries.

Corollary 1. *Let* $f(t) \doteqdot \psi(p)$ *and let* $t^{v-1/2}f(1/t^2)$ *be* $\bar{R}^m_n$. *Then* $t^{v-1/2}\psi(t^2/2)$ *will be* $\bar{R}^m_{n,v+1}$.

Corollary 2. *Let* $f(t) \doteqdot \psi(p)$ *and let* $t^{v-1/2}\psi(t^2/2)$ *be* $\bar{R}^m_{n,v+1}$. *Then* $t^{v-1/2}f(1/t^2)$ *will be* $\bar{R}^m_n$.

We have from (1.3), if $g_1(t) \doteqdot \varphi_1(p)$, then

$$p^{1-v}\varphi_1(1/p) \doteqdot t^{v/2}\int_0^\infty x^{-v/2}J_v(2\sqrt{tx})\,g_1(x)\,\mathrm{d}x\,, \quad R(v) > -1\,. \tag{2.18}$$

Using (2.18) for (2.16) and proceeding on the same lines as above we state the theorem as following.

Theorem 3. *Let* $f(t) \doteqdot \psi(p)$, $g(t) \doteqdot \varphi(p)$ *and* $x^{v-3/2}\varphi(x^2/2)$ *be the* $\overline{W}^m_n(x)$ *transform of* $y^{v-3/2}\psi(y^2)$. *Then* $y^{v-3/2}g(1/2y^2)$ *will be the* $\overline{W}^{m,v}_n(x)$ *transform of* $y^{v-3/2}\cdot f(1/2y^2)$, *provided* $x^{-v/2}f(x)$ *is continuous and absolutely integrable in* $(0, \infty)$ *and* $\left[g(t) - (1/\sqrt{2})\,t^{v/2-3/4}\int_0^\infty x^{v-3/2}f(1/x^2)\,\overline{W}^{m,v}_n(x/2\sqrt{t})\,\mathrm{d}x\right]$ *is a continuous function of t.*

Corollary. *Let* $f(t) \doteqdot \psi(p)$ *and* $x^{v-3/2}\psi(x^2)$ *be* $\bar{R}^m_n$. *Then* $x^{v-3/2}f(1/2x^2)$ *will be* $\bar{R}^{m,v}_n$, *provided that the conditions of the theorem are satisfied.*

Again on putting $F(t) = t^k\,\overline{W}^m_n(1/\sqrt{t})$ in (2.4) we have

$$y^k\int_0^\infty x^{-k-1}f(x)\,\overline{W}^m_n(\sqrt{x/y})\,\mathrm{d}x \doteqdot p^{-k}\int_0^\infty x^{k-1}\psi(x)\,\overline{W}^m_n(\sqrt{p/x})\,\mathrm{d}x\,. \tag{2.19}$$

Now using (2.19) for (2.12) and the relations (2.16), (2.18) and taking suitable values of k and proceeding as in the Theorem 2(a) we state the following theorems.

Theorem 4. *Let* $f(t) \doteqdot \psi(p)$, $g(t) \doteqdot \varphi(p)$ *and let* $x^{-v-1/2}g(x^2)$ *be the* $\overline{W}^m_n(x)$ *transform of* $y^{-v-1/2}f(y^2)$. *Then* $x^{-v-1/2}\varphi(1/2x^2)$ *will be the* $\overline{W}^{m,v+1}_n(x)$ *transform of* $y^{-v-1/2}\psi(1/2y^2)$, *provided* $x^{-v/2}f(x)$ *and* $x^{v/2-3/4}\psi(x)$ *are continuous and absolutely integrable in* $(0, \infty)$.

Corollary. *Let* $f(t) \doteqdot \psi(p)$ *and let* $x^{-v-1/2} f(x^2)$ *be* $\bar{R}_n^m$. *Then* $x^{-v-1/2}\psi(1/2x^2)$ *will be* $\bar{R}_n^{m,\,v+1}$, *provided that the conditions of the theorem are satisfied.*

Theorem 5. *Let* $f(t) \doteqdot \psi(p)$, $g(t) \doteqdot \varphi(p)$ *and let* $x^{1/2-v}\varphi(1/x^2)$ *be the* $\overline{W}_n^m(x)$ *transform of* $y^{1/2-v}\psi(1/y^2)$. *Then* $x^{1/2-v}\, g(x^2/2)$ *will be the* $\overline{W}_{n,v}^m(x)$ *transform of* $y^{1/2-v}\cdot f(y^2/2)$, *provided* $x^{v/2-1}\psi(x)$ *and* $x^{-v/2}f(x)$ *are continuous and absolutely integrable in* $(0, \infty)$.

Corollary 1. *Let* $f(t) \doteqdot \psi(p)$ *and let* $x^{1/2-v}\psi(1/x^2)$ *be* $\bar{R}_n^m$. *Then* $x^{1/2-v} f(x^2/2)$ *will be* $\bar{R}_{n,v}^m$, *provided that the conditions of the theorem are satisfies.*

3. - Examples.

(1) $$x^{v-1/2} f(1/x^2) = x^{u_1+u_2-1/2} \exp[-x^2/4]\, W_{(u_1+u_2)/4+(1-v_1)/2,\,(u_1-u_2)/4}(x^2/2)\,,$$

which is $R_{u_1,u_2}^{v_1}$ [4].

On taking $v = u_1/2 + u_2/2 - 2$ and $v_1 = u_1/2 + u_2/2$ we have

$$f(x) = \frac{1}{x} \exp[-1/4x]\, W_{1/2,\,(u_1-u_2)/4}(1/2x)$$

and

$$\psi(p) = \frac{2}{\sqrt{\pi}}\, p^{3/2} K_{(u_1-u_2)/4+1/2}\left(\sqrt{p/2}\right) K_{(u_1-u_2)/4-1/2}\left(\sqrt{p/2}\right).$$

Hence from the Corollary of the Theorem 2(a) we have

$$x^{(u_1+u_2+1)/2} K_{(u_1-u_2)/4+1/2}\,(x/2)\, K_{(u_1-u_2)/4-1/2}\,(x/2)$$

is $R_{u_1,u_2,u_1/2+u_2/2-1}^{u_1/2+u_2/2}$, $R(u_1+u_2) \geqslant 1$, $R(u_1, u_2) \geqslant -\frac{1}{2}$.

On taking $v = 2u_2 - u_1 - 1$ and $v_1 = u_2 + 1$ we have

$$f(x) = x^{-3(u_1/4-u_2/4)-1/2} \exp[-1/4x]\, W_{(u_1-u_2)/4,\,(u_1-u_2)/4}(1/2x)$$

and

$$\psi(p) = \frac{1}{\sqrt{\pi}}\, 2^{(u_1-u_2+2)/4}\, p^{(u_1-u_2)/4+1} \left[K_{(u_1-u_2)/2}\left(\sqrt{p/2}\right)\right]^2.$$

Hence from the Corollary of the Theorem 2(a) we have

$$x^{u_2+1/2}[K_{(u_1-u_2)/2}(x/2)]^2 \quad \text{is} \quad R^{u_2+1}_{u_1, u_2, 2u_2-u_1},$$

$R(u_1, u_2, 2u_2-u_1) \geqslant -\frac{1}{2}$, $R(3u_1-u) > -2$, $R(3u_2-u_1) > -2$.

Let

$$x^{v-1/2}f(1/x^2) = x^{u_1/2+u_2/2-1} \exp[x^2/4]\, W_{-\lambda/2+(u_1+u_2)/4,\,(u_1-u_2)/4}(x^2/2) \tag{2}$$

which is $R^{\lambda}_{u_1, u_2}$ [4]. On taking $v = -\lambda-1$ we have

$$f(x) = x^{-\lambda/2-u_1/2-u_2/2-1/2} \exp[1/4x]\, W_{-(\lambda/2)+(u_1+u_2+2)/4,\,(u_1-u_2)/4}(1/2x)$$

and

$$\psi(p) = 2^{u_1/2+u_2/2+\lambda+3/2} p^{u_1/4+u_2/4+\lambda/2+1} S_{-\lambda-1-u_1/2-u_2/2,\, u_1/2-u_2/2}(\sqrt{2p}).$$

Hence from the Corollary of the Theorem 2(a) we have

$$x^{u_1/2+u_2/2+1/2} S_{-\lambda-1-u_1/2-u_2/2,\, u_1/2-u_2/2}(x) \quad \text{is} \quad R^{\lambda}_{-\lambda, u_1, u_2};$$

$\frac{1}{2} \geqslant R(\lambda) \geqslant -\frac{1}{2}$, $R(u_1, u_2) \geqslant -\frac{1}{2}$, $R(\lambda-u_1) < 2$, $R(\lambda-u_2) < 2$.

Let

$$f(x) = x^{u/2+v/2+1/2}\, {}_2F_3\left[\begin{matrix} \frac{u+\theta}{2}+1, \frac{u+\varphi}{2}+1; \\ \frac{u+\lambda}{2}+1, \frac{u+\delta}{2}+1, \frac{u+\eta}{2}+1; \end{matrix} \; -x/4\right]. \tag{3}$$

Then

$$x^{-v-1/2}f(x^2) = x^{u+1/2}\, {}_2F_3\left[\begin{matrix} \frac{u+\theta}{2}+1, \frac{u+\varphi}{2}+1; \\ \frac{u+\lambda}{2}+1, \frac{u+\delta}{2}+1, \frac{u+\eta}{2}+1; \end{matrix} \; -x^2/4\right],$$

which is $R^{\theta,\varphi}_{u,\lambda,\delta,\eta,}$ [7], $R(\eta-u+\delta+\lambda-\theta-\varphi) > 2$, $R(u, \eta, \delta, \lambda, \theta, \varphi) \geqslant -\frac{1}{2}$, and

$$\psi(p) = \Gamma\left(\frac{u+v+3}{2}\right) p^{-u/2-v/2-1/2} {}_3F_3\left[\begin{matrix} \frac{u+\theta}{2}+1, \frac{u+\varphi}{2}+1, \frac{u+v+3}{2}; \\ \frac{u+\lambda}{2}+1, \frac{u+\delta}{2}+1, \frac{u+\eta}{2}+1; \end{matrix} -1/4p\right],$$

$R(u+v) > -3, \ R(p) > 0.$

Hence from the Corollary of the Theorem 4 we have

$$x^{u+1/2} {}_3F_3\left[\begin{matrix} \frac{u+\theta}{2}+1, \frac{u+\varphi}{2}+1, \frac{u+v+3}{2}; \\ \frac{u+\lambda}{2}+1, \frac{u+\delta}{2}+1, \frac{u+\eta}{2}+1; \end{matrix} -x^2/2\right]$$

is $R^{\theta, \varphi, v+1}_{u, \lambda \delta, \eta,}$, $R(v) > -\frac{3}{2}$.

Let

$$f(x) = x^{u/2+v/2} {}_3F_4\left[\begin{matrix} \frac{u+\theta}{2}+1, \frac{u+\varphi}{2}+1, \frac{u+\xi}{2}+1; \\ \frac{u+\lambda}{2}+1, \frac{u+\delta}{2}+1, \frac{u+\eta}{2}+1, \frac{u+v}{2}+1; \end{matrix} -x/2\right]. \tag{4}$$

Then

$$\psi(x) = \Gamma\left(\frac{u+v}{2}+1\right) x^{-u/2-v/2} {}_3F_3\left[\begin{matrix} \frac{u+\theta}{2}+1, \frac{u+\varphi}{2}+1, \frac{u+\xi}{2}+1; \\ \frac{u+\lambda}{2}+1, \frac{u+\delta}{2}+1, \frac{u+\eta}{2}+1; \end{matrix} -x/2\right],$$

$R(u+v) > 0$, and

$$x^{1/2-v} \psi(1/x^2) = x^{u+1/2} {}_3F_3\left[\begin{matrix} \frac{u+\theta}{2}+1, \frac{u+\varphi}{2}+1, \frac{u+\xi}{2}+1; \\ \frac{u+\lambda}{2}+1, \frac{u+\delta}{2}+1, \frac{u+\eta}{2}+1; \end{matrix} -x^2/2\right],$$

which is $R^{\theta, \varphi, \xi}_{u, \lambda, \delta, \eta}$, $R(u, \eta, \delta, \lambda, \theta, \varphi, \xi) \geqslant -\frac{1}{2}$.

Hence from the Corollary of the Theorem 5 we have

$$x^{u+1/2}\,{}_3F_4\left[\begin{matrix}\frac{u+\theta}{2}+1,\ \frac{u+\varphi}{2}+1,\ \frac{u+\xi}{2}+1; \\ \frac{u+\lambda}{2}+1,\ \frac{u+\delta}{2}+1,\ \frac{u+\eta}{2}+1,\ \frac{u+v}{2}+1;\end{matrix}\ -x^2/4\right]$$

is $R^{\theta,\varphi,\xi}_{u,\lambda\,\delta,\eta,v}$, $R(v) \geqslant -\frac{1}{2}$.

My best tansks are due to Dr. S. C. MITRA for his helpful suggestions in the preparation of this paper.

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Summary.

The object of this paper is to prove certain theorems on generalised Henkel transform and self-reciprocal functions. Certain new self-reciprocal functions are obtained by the help of these theorems.

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