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Some theorems on integral transforms. (**)

1. - Introduction.

In this paper first we establish a general theorem concerning integral transforms introduced by Meijer and Varma. Later on we evaluate some new and interesting infinite integrals involving the H-function with the help of some of the special cases of the theorem. Next we prove a theorem showing close relationship between generalized Stieltjes transform and Mainra transform. Several results obtained earlier follow as particular cases of our findings.

The H-function.

The H- function will be defined and represented as follows ([1] p. 239):

$$(1.1)\qquad H_{p,q}^{m,n}\left[z\left|\begin{matrix}(a_1,\alpha_1),\dots,(a_p,\alpha_p)\\(b_1,\beta_1),\dots,(b_q,\beta_q)\end{matrix}\right.\right]=$$

$$=\frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j-\beta_j\xi)\prod_{j=1}^{n}\Gamma(1-a_j+\alpha_j\xi)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+\beta_j\xi)\prod_{j=n+1}^{p}\Gamma(a_j-\alpha_j\xi)}\,z^{\xi}\,d\xi\,,$$

where z is not equal to zero and an empty product is interpreted as 1; p, q, n and m are integers satisfying: $1\leqslant m\leqslant q$; $0\leqslant n\leqslant p$; α_j $(j=1,\dots,p)$, β_j $(j=1,\dots,q)$ are positive numbers, and a_j $(j=1,\dots,p)$, b_j $(j=1,\dots,q)$ are

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complex numbers such that no pole of $\Gamma(b_h - \beta_h \xi)$ $(h = 1, \ldots, m)$ coincides with any pole of $\Gamma(1 - a_i + \alpha_i \xi)$ $(i = 1, \ldots, n)$ i.e.

$$\alpha_i(b_h + \nu) \neq \beta_h(a_i - \eta - 1) \qquad (\nu, \eta = 0, 1, \ldots; h = 1, \ldots, m; i = 1, \ldots, n)\,. \tag{1.2}$$

Furhter the contour L runs from $\sigma - i\infty$ to $\sigma + i\infty$ such that the points

$$\xi = \frac{b_h + \nu}{\beta_h} \qquad (h = 1, \ldots, m; \nu = 0, 1, 2, \ldots),$$

which are poles of $\Gamma(b_h - \beta_h \xi)$, lie to the right, and the points

$$\xi = \frac{a_i - \eta - 1}{\alpha_i} \qquad (i = 1, \ldots, n; \eta = 0, 1, 2, \ldots),$$

which are poles of $\Gamma(1 - a_i + \alpha_i \xi)$, lie to the left of L. Such a contour is possible on account of (1.2). These assumptions for the H-function will be adhered to throughout this paper.

Explanations of the symbols used.

Hereinafter a function which is either continuous or sectionally continuous and whose orders for small x and large x are as follows

$$f(x) \begin{cases} = O(x^{\alpha}) & \text{for small } x \\ = O\big(x^{\beta} \exp(ax)\big) & \text{for large } x\,, \end{cases}$$

where α, β and a are real or complex will be represented symbolically as $f(x) \in$ $\in A(\alpha, \beta, a)$.

Also

a) N, S will always denote positive integers s;

b) $\Delta(N, \alpha)$ will stand for α/N, $(\alpha + 1)/N, \ldots, (\alpha + N - 1)/N$;

c) $(\alpha \pm \beta, \sigma)$ for the pairs $(\alpha + \beta, \sigma)$, $(\alpha - \beta, \sigma)$;

d) $\Delta(N, \alpha \pm \beta)$ for $\Delta(N, \alpha + \beta)$, $(N, \alpha - \beta)$.

In the next section we shall establish relations concerning the transforms defined below:

Verma transform [**10**] given by

$$V\{f(x); k, r; s\} = s\int_0^{\infty} (sx)^{r-\frac{1}{2}} \exp\left(-\tfrac{1}{2}sx\right) W_{k,r}(sx)\, f(x)\, dx; \tag{1.3}$$

Mejer transform ([**6**] p. 209) defined as

$$M\{f(x);\,\nu,\,s\}=\sqrt{\frac{2}{\pi}}\,s\int\limits_0^\infty (sx)^{\frac12}\,K_\nu(sx)\,f(x)\,\mathrm{d}(x)\,; \tag{1.4}$$

Laplace transform represented as

$$\varphi(s)=\mathscr{L}\{f(x);\,s\}=s\int\limits_0^\infty \exp\,[-sx]\,f(x)\,\mathrm{d}x\,. \tag{1.5}$$

Both the transforms given by (1.3) and (1.4) reduce to (1.5) if we put $k+r=\frac12$ and $\nu=\pm\frac12$ in them respectively.

We shall also represent (1.5) symbolically as $\phi(s)\doteqdot f(x)$.

2. – Theorem 1. *If*

$$f(s)=M\{g(x)\phi(x);\,\nu;\,s\} \tag{2.1}$$

and

$$s^l\varphi(s^\sigma)=V\{h(x);\,k,\,r;\,s\}\,, \tag{2.2}$$

then

$$f(s)=\left(\frac{2}{\pi}\right)^{\frac12} s^{\frac32}\,\sigma\int\limits_0^\infty \frac{h(t)}{t}\,V\{x^{\frac32\sigma-l}g(x^\sigma)K_\nu(sx^\sigma);\,k,\,r;\,t\}\,\mathrm{d}t\,, \tag{2.3}$$

provided that: $h(x)\in A(\alpha,\beta,a)$ $[R(a)\leqslant 0]$; $g(x)\in A(\gamma,\delta,b)$, $x^{\frac12}g(x)\phi(x)K_\nu(sx)\in L(0,\infty)$; $R(s)>R(b)$; $R(\alpha+2r+1)>0$; $R(\alpha+1)>0$; $\sigma>0$; $R(r\pm r\pm\pm\nu\sigma+\frac32\sigma-l+\gamma\sigma+1)>0$.

Proof. Interpreting (2.2) with the help of (1.3), we get

$$\varphi(s^\sigma)=s^{1-l}\int\limits_0^\infty (st)^{r-\frac12}\exp\,(-\tfrac12 st]\,W_{k,r}(st)\,h(t)\,\mathrm{d}t\,, \tag{2.4}$$

where $R(s)>R(a)$, $\sigma>0$ and $R(\alpha+r\pm r+1)>0$, therefore

$$\varphi(x)=x^{1-l/\sigma}\int\limits_0^\infty (x^{1/\sigma}t)^{r-\frac12}\exp\,(-\tfrac12 tx^{1/\sigma})\,W_{k,r}(x^{1/\sigma}t)\,h(t)\,\mathrm{d}t\,.$$

Using the above value of $\phi(x)$ in (2.1), we obtain

$$(2.5)\qquad f(s)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}} s\int_0^\infty (sx)^{\frac{1}{2}}K_\nu(sx)\,g(x)\times$$

$$\times[\,x^{1-l/\sigma}\int_0^\infty(tx^{1/\sigma})^{r-\frac{1}{2}}\exp\left(-\tfrac{1}{2}tx^{1/\sigma}\right)W_{k,r}(tx^{1/\sigma})h(t)\,\mathrm{d}t]\,\mathrm{d}x\,.$$

On interchanging the order of intergation in (2.5), we get

$$(2.6)\qquad f(s)=\left(\frac{2}{\pi}\right)^{\frac{1}{2}} s\int_0^\infty t^{r-\frac{1}{2}}h(t)\times$$

$$\times[\int_0^\infty(sx)^{\frac{1}{2}}K_\nu(sx)\,x^{1/\sigma(r-l+\frac{1}{2})}\exp\left(-\tfrac{1}{2}tx^{1/\sigma}\right)W_{k,r}(tx^{1/\sigma})\,g(x)\,\mathrm{d}x]\,\mathrm{d}t\,.$$

Putting $x^{1/\sigma}=y$ in the inner integral of (2.6) and interpreting the result thus obtained with the help of (1.3), we arrive at the required result.

All that remains now is to justify the interchange of the order of integration in (2.5). For this we notice that since $h(t)\in A(\alpha,\beta,a)$, we have on taking

$$U\equiv t^{r-\frac{1}{2}}\exp\left(-\tfrac{1}{2}tx^{1/\sigma}\right)W_{k,r}(tx^{1/\sigma})\,h(t)\,,$$

$$U\begin{cases}\sim t^{r\pm r+\alpha} & \text{for small } t\\[6pt] \sim t^{r+k+\beta-\frac{1}{2}}\exp\left(-t(x^{1/\sigma}-a)\right) & \text{for large } t\,.\end{cases}$$

Thus t-integral in (2.5) is absolutely convergent when $R(\alpha+r\pm r+1)>0$ and $R(a)\leqslant 0$.
Also on taking

$$U\equiv x^{1/\sigma(r+\frac{1}{2}\sigma-l+\frac{1}{2})}\exp\left(-\tfrac{1}{2}tx^{1/\sigma}\right)W_{k,r}(tx^{1/\sigma})\,K_\nu(sx)\,g(x)$$

$$U\begin{cases}\sim x^{1/\sigma(r\pm r+\frac{1}{2}\sigma-l\pm\nu\sigma+\gamma\sigma+1)} & \text{for small } x\\[6pt] \sim x^{1/\sigma(r+k+\frac{1}{2}-l+\delta\sigma)}\exp\left(-tx^{1/\sigma}\right)\exp\left(-x(s-b)\right) & \text{for large } x\,.\end{cases}$$

So x-integral in (2.5) is absolutely convergent when $R(s)>R(b)$, and $R(r\pm r+1+\frac{3}{2}\sigma-l\pm\sigma\nu+\gamma\sigma)>0$.

Finally $f(s)$ exists when $x^{\frac{1}{2}}g(x)\varphi(x)K_\nu(sx)\in L(0,\infty)$. So the inversion of the order of integration in (2.5) is justified under the conditions stated with the theorem by virtue of De La Vallée Poussin's theorem ([**2**] p. 504).

Theorem 1 (a). *For* $k + r = \frac{1}{2}$, *Theorem* 1 *takes the following form*: *If*

$$f(s) = M\{g(x)\phi(x);\, \nu;\, s\}$$

and

$$s^{l}\varphi(s^{\sigma}) = \mathscr{L}\{h(x);\, s\}\,,$$

then

$$(2.7) \qquad f(s) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} s^{\frac{3}{2}}\sigma \int_0^\infty \frac{h(t)}{t} \times \mathscr{L}\{x^{\frac{3}{2}\sigma - l} K_\nu(sx^{\sigma})\, g(x^{\sigma});\, t\}\, dt\,,$$

where: $h(x) \in A(\alpha, \beta, a)$ $[R(a) \leqslant 0]$, $R(\alpha + 1) > 0$, $\sigma > 0, g(x) \in A(\gamma, \delta, b)$, $R(s) > \max\{R(b), 0\}$, $x^{\frac{1}{2}} g(x)\phi(x)K_\nu(sx) \in L(0, \infty)$ and $R(1 - l + \gamma\sigma + \frac{3}{2}\sigma \pm \nu\sigma) > 0$.

If we take $\sigma = l = 1$ in Theorem 1 (a), we get a theorem obtained by Sharma ([**9**] p. 362).

Theorem 1 (b). *The well known property of Laplace transform* ([**4**] p. 129) *reduces Theorem* 1 (a) *to the following form, if we put* $\sigma = 1$ *and* $\nu = \pm\frac{1}{2}$ *therein*: *If*

$$f(s) = \mathscr{L}\{g(x)\varphi(x);\, s\}\,, \qquad s^{l}\varphi(s) = \mathscr{L}\{h(x);\, s\}\,,$$

then

$$(2.8) \qquad f(s) = s\int_0^\infty \frac{h(t)}{s + t}\, \mathscr{L}\{x^{1-l} g(x);\, s + t\}\, dt\,,$$

where

$$h(x) \in A(\alpha, \beta, a) \quad [R(a) \leqslant 0]\,, \qquad g(x) \in A(\gamma, \delta, b)\,,$$

$R(s) > \max[R(b), 0]$, $R(\alpha + 1) > 0$, $R(2 - l + \gamma) > 0$ *and* $g(x)\varphi(x)\exp(-sx) \in L(0, \infty)$.

Special cases of Theorem 1 (b).

Corollary 1. *If we take* $g(x) = \exp(-ax^{\sigma})$, $\phi(x) = \exp(ax^{\sigma})\psi(x)$, $h(x) = \theta(x)$ *in Theorem* 1 (b) *and replace* l *by* $2 - l$ *therein, we get the following result. If*

$$f(s) = \psi(s)\,, \qquad s^{2-l}\exp(as^{c})\psi(s) = \theta(x)\,,$$

then

$$(2.9)\qquad f(s) = sa^{-l/\sigma}\int_0^\infty \theta(x) H_{1,1}^{1,1}\left[\frac{1}{a}(x+s)^\sigma \middle| \begin{matrix}(\sigma - l/\sigma, 1)\\ (0, \sigma)\end{matrix}\right] dx\,,$$

where $\theta(x)\in A(\alpha, \beta, b)$ $[R(b)\leqslant 0]$, $R(\alpha+1)>0$, $R(l)>0$, $R(\beta+1-l)>0$ *and the following set of conditions is satisfied:*

i) $0<\sigma<1$, $R(s)>0$;
ii) $\sigma=1$, $R(s)>\max[R(-a), 0]$;
iii) $\sigma>1$, $R(a)>0$ *and* $R(s)>0$.

Proof. We would like to sketch an outline of the proof of the above corollary, though it is a particular case of Theorem 1 (b) because the function $\exp(-ax^\sigma)$ does not belong to $A(\gamma, \delta, b)$ as required by $g(x)$ of the Theorem 1 (b). We have [4]

$$(2.10)\qquad x^{l-1}\exp(-\alpha x^\sigma) \doteqdot a^{-1/\sigma(l-1)} H_{1,1}^{1,1}\left[as^{-\sigma}\middle| \begin{matrix}(0, \sigma)\\ (l-1/\sigma, 1)\end{matrix}\right],$$

where $R(l)>0$ and the following set of conditions is satisfied:

i) $0<\sigma<1$, $R(s)>0$;
ii) $\sigma=1$, $R(s)>R(-a)$;
iii) $\sigma>1$, $R(a)>0$.

Using (2.10) and proceeding in a manner similar to that of Theorem 1 we arrive at the required corollary.

Particular case. If we replace s by p in Corollary 1 given above and then substitute $\sigma = N/S$ there, we arrive at a theorem given by Saxena ([7] p. 43).

Example. If we take

$$\psi(x) = x^{-c\sigma}\exp(-\tfrac{1}{2}x^\sigma)\, W_{k,r}(x^\sigma)$$

in Corollary 1, we have [4]

$$(2.11)\qquad \psi(x) \doteqdot H_{2,2}^{2,1}\left[s^{-\sigma}\middle| \begin{matrix}(0, \sigma),\ (1-c-k, 1)\\ (\tfrac{1}{2}-c\pm r, 1)\end{matrix}\right] = f(s)\,,$$

where $R(s)>0$ and $R(1-c\sigma+\sigma/2\pm r\sigma)>0$.

Also, on account of a well known identity for the G-function ([3] p. 221, equation (69)), we have

$$s^{2-l}\exp(s^{\sigma})\,\psi(s)=\frac{s^{2-l-c\sigma}}{\Gamma(\frac{1}{2}-k\pm r)}\,G^{1,2}_{2,1}\left[s^{-\sigma}\middle|\begin{matrix}\frac{1}{2}\pm r\\ -k\end{matrix}\right].$$

Therefore [4] yields

$$s^{2-l}\exp(s\sigma)\,\psi(s)\doteqdot\frac{x^{l+c\sigma-2}}{\Gamma(\frac{1}{2}-k\pm r)}\,H^{1,2}_{2,2}\left[x^{\sigma}\middle|\begin{matrix}(\frac{1}{2}\pm r,1)\\ (-k,1),\ (2-l-c\sigma,\sigma)\end{matrix}\right],\tag{2.12}$$

where $R(s)>0$, $1<\sigma<3$ and $R(l+c\sigma-1)>0$.

Applying Corollary 1 in the values of $f(s)$ and $\theta(x)$ thus obtained, we get the following integral:

$$\int_0^\infty x^{l+c\sigma-2}H^{1,1}_{1,1}\left[(x+s)^{\sigma}\middle|\begin{matrix}\sigma-l/\sigma,1\\ (0,\sigma)\end{matrix}\right]H^{1,2}_{2,2}\left[x^{\sigma}\middle|\begin{matrix}(\frac{1}{2}\pm r,1)\\ (-k,1),\ (2-l-c\sigma,\sigma)\end{matrix}\right]dx=\tag{2.13}$$

$$=\frac{1}{s}\Gamma(\tfrac{1}{2}-k\pm r)H^{1,2}_{2,2}\left[s^{\sigma}\middle|\begin{matrix}(\frac{1}{2}+c\pm r,1)\\ (1,\sigma),\ (c+k,1)\end{matrix}\right],$$

where $1<\sigma<3$, $R(s)>0$, $R(l+c\sigma-1-\sigma k)>0$, $R(l)>0$, and $R(2+\sigma-2c\sigma\pm 2r\sigma)>0$.

Particular case. If we replace s by $pN(S)^{-s/N}$, x by $tNS^{-s/N}$ and substitute $\sigma=N/S$ in (2.13), we get an integral given by Saxena ([7] p. 48).

Corollary 2. *Again on taking $g(x)=\exp(-ax^{-\sigma})$, $\phi(x)=\exp(ax^{-\sigma})\psi(x)$, $h(x)=\theta(x)$ in Theorem* 1 (b) *and then replacing l by $2-l$ there, we get the following result: If*

$$f(s)\doteqdot\psi(x)\,,\quad \textit{and}\quad s^{2-l}\exp(as^{-\sigma})\,\psi(s)\doteqdot\theta(x)\,,$$

then

$$f(s)=sa^{l/\sigma}\int_0^\infty\theta(x)H^{2,0}_{0,2}\left[a(x+s)^{\sigma}\middle|(0,\sigma),\left(-\frac{l}{\sigma},1\right)\right]dx\,,\tag{2.14}$$

where $\theta(x)\in A(\alpha,\beta,b)$ $[R(b)\leqslant 0]$, $\sigma>0$, $R(s)>0$, $R(a)>0$ and $R(\alpha+1)>0$.

Proof. Here also we sketch the proof on account of the reasions mentioned with Corollary 1.

We have [4]

$$x^{l-1}\exp(-ax^{-\sigma}) \doteqdot a^{l-1/\sigma} H^{2,0}_{0,2}\left[as^{\sigma}\middle|(1,\sigma),\left(\frac{1-l}{\sigma},1\right)\right], \tag{2.15}$$

where $\sigma > 0$, $R(s) > 0$ and $R(a) > 0$.

With the help of (2.15), and preceeding as indicated in Theorem 1, we get the required result.

Particular case. If we replace s by p and then put $\sigma = N/S$ in Corollary 2, we get another theorem given by Saxena ([7] p. 49).

Example. On taking

$$\psi(x) = x^{-c\sigma}\exp(-\tfrac{1}{2}x^{-\sigma})\,W_{k,r}(x^{-\sigma}),$$

we have [4]

$$\psi(x) \doteqdot H^{3,0}_{1,3}\left[s^{\sigma}\middle|\begin{matrix}(1+c-k,1)\\(1,\sigma),\ (\frac{1}{2}+c\pm r,1)\end{matrix}\right] = f(s), \tag{2.16}$$

where $\sigma > 0$ and $R(s) > 0$.

Also, on account of a well known formula for the G-function ([3] p. 221),

$$s^{2-l}\exp(s^{-\sigma})\psi(s) = s^{2-l-c\sigma}G^{2,1}_{1,2}\left[s^{-\sigma}\middle|\begin{matrix}k+1\\\frac{1}{2}\pm r\end{matrix}\right].$$

Therefore [4] gives

$$\begin{aligned} s^{2-l}\exp(s^{-\sigma})\psi(s) &\doteqdot \frac{x^{l+c\sigma-2}}{\Gamma(\frac{1}{2}-k\pm r)}H^{2,1}_{1,3}\left[x^{\sigma}\middle|\begin{matrix}(k+1,1)\\(\frac{1}{2}\pm r,1),\ (2-l-c\sigma,\sigma)\end{matrix}\right], \\ &= \theta(x), \end{aligned} \tag{2.17}$$

where $\quad R(s) > 0, \quad 0 < \sigma < 3, \quad R\left(l + c\sigma - 1 + \frac{\sigma}{2} \pm r\sigma\right) > 0.$

Applying Corollary 2 to these values of $\theta(x)$ and $f(s)$, we get the following integral

$$(2.18)\quad \int_0^\infty x^{l+c\sigma-2} H^{2,1}_{1,3}\left[x^\sigma \middle| \begin{matrix} (k+1,1) \\ (\tfrac{1}{2}\pm r,1),\ (2-l-c\sigma,\sigma) \end{matrix}\right] H^{2,0}_{0,2}\left[(x+s)^\sigma \middle| (0,\sigma),\left(-\frac{l}{\sigma},1\right)\right] dx$$

$$= \frac{1}{s}\,\Gamma(\tfrac{1}{2}-k\pm r)\,H^{3,0}_{1,3}\left[s^\sigma \middle| \begin{matrix} (c-k+1,1) \\ (1,\sigma),\ (\tfrac{1}{2}+c\pm r,1) \end{matrix}\right],$$

where $R(s)>0$, $0<\sigma<3$ and $R(l+c\sigma-1+\sigma/2\pm r\sigma)>0$.

On taking $\sigma = N/S$ in (2.18) and replacing x by $tNS^{s/N}$, S by $pNS^{s/N}$ there, we easily get an integral obtained by Saxena ([7] p. 51).

Corollary 3. *If we substitute* $g(x)=(1+x^\sigma)^{-\alpha}$, $\phi(x)=(1+x^\sigma)^\alpha\psi(x)$ *and,* $h(x)=\theta(x)$ *in Theorem* 1 (b) *and then replace* l *by* $2-l$ *there, we get the following corollary: If*

$$f(s) \doteqdot \psi(x)\,, \qquad s^{2-l}(1+s^\sigma)^\alpha\psi(s) \doteqdot \theta(x)\,,$$

then

$$(2.19)\qquad f(s) = \frac{s}{\Gamma(\alpha)}\int_0^\infty \theta(x) H^{2,1}_{1,2}\left[(x+s)^\sigma \middle| \begin{matrix} (\sigma-l/\sigma,1) \\ (0,\sigma),\ (\alpha-l/\sigma,1) \end{matrix}\right] dx\,,$$

where $\theta(x)\in A(\alpha',\beta,a)$ $[R(a)\leqslant 0]$, $R(s)>0$, $\sigma>0$, $R(l)>0$, $R(\alpha'+1)>0$ *and* $R(\beta+1)<0$.

Proof. We have [4]

$$(2.20)\qquad x^{l-1}(1+x^\sigma)^{-\alpha} \doteqdot \frac{s^{1-l}}{\Gamma(\alpha)} H^{1,2}_{2,1}\left[s^{-\sigma} \middle| \begin{matrix} (1-l,\sigma),\ (1-\alpha,1) \\ (0,1) \end{matrix}\right],$$

where $R(s)>0$ and $R(l)>0$. With the help of (2.20), Theorem 1 (b) reduces to the required corollary forthwith.

Particular case. Replacing s by p in Corollary 3 and then putting $\sigma = N/S$ in it, we get a known result ([7] p. 56 equation No. 62).

Example. On taking

$$s^{2-l}(1+s^\sigma)^{\gamma-\beta-\alpha}\psi(s) = s^c\,{}_2F_1(\alpha,\beta,\gamma;-s^\sigma)$$

we easily have [4]

$$(2.21)\qquad \theta(x)=\frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\beta)}\,x^{-c}H_{2,3}^{2,1}\left[x^{\sigma}\middle|\begin{matrix}(1,1),\ (\gamma,1)\\(\sigma,1),\ (\beta,1),\ (c,\sigma)\end{matrix}\right],$$

where $R(s)>0$, $R(\alpha\sigma-c+1)>0$ and $R(\beta\sigma-c+1)>0$.

Again [4] gives

$$(2.22)\qquad f(s)=\frac{\Gamma(\gamma)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}\,s^{2-c-l}H_{3,2}^{1,3}\left[s^{-\sigma}\middle|\begin{matrix}(2-l-c,\sigma),\ (\alpha-r+1,1),\ (\beta-r+1,1)\\(0,1),\ (1-\gamma,1)\end{matrix}\right],$$

where $R(s)>0$ and $R(l+c-1)>0$.

Putting these values of $\theta(x)$ and $f(s)$ in Corollary 3, we get the integral

$$(2.23)\qquad \int_0^\infty x^{-c}H_{2,3}^{2,1}\left[x^{\sigma}\middle|\begin{matrix}(1,1),\ (\gamma,1)\\(\alpha,1),\ (\beta,1),\ (c,\sigma)\end{matrix}\right]H_{1,2}^{2,1}\left[(x+s)^{\sigma}\middle|\begin{matrix}(1-l/\sigma,1)\\(0,\sigma),\ (\gamma-\alpha-\beta-l/\sigma,1)\end{matrix}\right]dx=$$

$$=\frac{\Gamma(\gamma-\alpha-\beta)\,\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)}\,s^{1-c-l}\times H_{3,2}^{1,3}\left[s^{-\sigma}\middle|\begin{matrix}(2-l-c,\sigma),\ (1+\alpha-\gamma,1),\ (1+\beta-\gamma,1)\\(0,1),\ (1-\gamma,1)\end{matrix}\right],$$

where $R(\alpha\sigma-c+1)>0$, $R(\beta\sigma-c+1)>0$, $R(l+c-1)>0$.

On taking $\sigma=N/S$ in (2.23), we get the following integral

$$(2.24)\qquad \int_0^\infty x^{-c}G_{2S,2S+N}^{2S,S}\left[\left(\frac{x}{N}\right)^{N}\middle|\begin{matrix}\Delta(S,1),\ \Delta(S,\gamma)\\\Delta(S,\alpha),\ \Delta(S,\beta),\ \Delta(N,c)\end{matrix}\right]\times$$

$$\times G_{S,S+N}^{S+N,S}\left[\left(\frac{x+s}{N}\right)^{N}\middle|\begin{matrix}\Delta(S,1-lS/N)\\\Delta(N,0),\ \Delta(S,\gamma-\alpha-\beta-lS/N)\end{matrix}\right]dx$$

$$=\frac{\Gamma(\gamma-\alpha-\beta)\,\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)}\,s^{1-c-l}N^{l-\frac{1}{2}}S^{\gamma-\alpha-\beta}(2\pi)^{\frac{1}{2}(2S-N-1)}\times$$

$$\times G_{N+2S,2S}^{S,N+2S}\left[\left(\frac{N}{s}\right)^{N}\middle|\begin{matrix}\Delta(N,2-l-c),\ \Delta(S,1+\alpha-\gamma),\ \Delta(S,1+\beta-\gamma)\\\Delta(S,0),\ \Delta(S,1-\gamma)\end{matrix}\right],$$

where $R(l+c-1)>0$, $R(\alpha N-cS+S)>0$, and $R(\beta N-cS+S)>0$. (2.24) was also given earlier by Saxena ([7] p. 58). There seems to be some misprint in the powers of N and S in his result.

3. – In this section we shall establish a theorem concerning the transforms defined below:

(a) Generalized Stieltjes transform given by

$$S\{f(x);\,\alpha;\,s\} = s\int_0^\infty (s+x)^{-\alpha} f(x)\,\mathrm{d}x; \tag{3.1}$$

(b) Mainra transform ([**5**] p. 24) defined and represented as

$$\begin{aligned} W\{f(x);\,\eta+\tfrac{1}{2};\,k+\tfrac{1}{2},r;\,s\} &= \\ &= s\int_0^\infty (sx)^{-\eta-\frac{1}{2}} \exp\left(-\tfrac{1}{2}sx\right) W_{k+\frac{1}{2},r}(sx)\,f(x)\,\mathrm{d}x\,. \end{aligned} \tag{3.2}$$

Theorem 2. *If*

$$\psi(s) = S\{x^{c+l}f(x);\,\alpha;\,s\}$$

and

$$s^{(l+1)\sigma}f(s^\sigma) = W\{\varphi(x);\,\eta+\tfrac{1}{2};\,k+\tfrac{1}{2},r;\,s\},$$

then

$$\begin{aligned} \psi(s) &= \frac{\sigma}{\Gamma(\alpha)}\, s^{c+1-\alpha+1/\sigma} \times \\ &\times \int_0^\infty \varphi(x)\, H^{3,1}_{2,3}\left[sx^\sigma \left| \begin{matrix} (1-c-1/\sigma,\,1),\ (-k-\eta,\,\sigma) \\ (-\eta\pm r,\,\sigma),\ (\alpha-c-1/\sigma,\,1) \end{matrix} \right.\right] \mathrm{d}x\,, \end{aligned} \tag{3.3}$$

where $\phi(x)\in A(\gamma,\beta,a)$ $\{R(a)\leqslant 0\}$, $\sigma>0$, $R(s)>0$, $R(c\sigma-\beta)>0$, $R(-\eta\pm r+$ $+\gamma+1)>0$, $R(\gamma+\sigma\alpha-c\sigma)>0$, *and* $R(c\sigma-\eta\pm r+1)>0$.

Proof. We have with the help of (3.2)

$$s^{(l+1)\sigma}f(s^\sigma) = s\int_0^\infty (st)^{-\eta-\frac{1}{2}} \exp\left(-\tfrac{1}{2}st\right) W_{k+\frac{1}{2},r}(st)\,\varphi(t)\,\mathrm{d}t\,,$$

where $R(s)>0$ and $R(-\eta\pm r+\gamma+1)>0$; therefore

$$x^{c+l}f(x) = x^{c-1+1/\sigma}\int_0^\infty (tx^{1/\sigma})^{-\eta-\frac{1}{2}} \exp -\frac{t}{2}\,x^{1/\sigma}\, W_{k+\frac{1}{2},r}(tx^{1/\sigma})\,\varphi(t)\mathrm{d}t. \tag{3.4}$$

Also, with the help of (3.1) and (3.4) we have

$$(3.5)\qquad \psi(s) = s\int_0^\infty (s+x)^{-\alpha}\left\{x^{c-1+1/\sigma}\int_0^\infty (tx^{1/\sigma})^{-\eta-\frac{1}{2}}\times\right.$$
$$\left.\times\exp-\frac{t}{2}\,x^{1/\sigma}\,W_{k+\frac{1}{2},r}(tx^{1/\sigma})\,\varphi(t)\,\mathrm{d}t\right\}\mathrm{d}x.$$

On interchanging the order of integration in (3.5) we get

$$(3.6)\qquad \psi(s) = s\int_0^\infty t^{-\eta-\frac{1}{2}}\varphi(t)\left[\int_0^\infty (s+x)^{-\alpha}\times\right.$$
$$\left.\times x^{1/\sigma(c\sigma-\eta+\frac{1}{2}-\sigma)}\exp-\frac{t}{2}\,x^{1/\sigma}\,W_{k+\frac{1}{2},r}(tx^{1/\sigma})\,\mathrm{d}x\right]\mathrm{d}t\,.$$

Putting $x^{1/\sigma}=\xi$ in the equation (3.6) and evaluating the ξ-integral thus obtained with the help of the following formula [4]

$$(3.7)\qquad W\{x^l(a+x^\sigma)^{-\alpha};\,\eta+\tfrac{1}{2};\,k+\tfrac{1}{2},r;\,s\} =$$
$$= \frac{a^{-\alpha}s^{-l}}{\Gamma(\alpha)}\,H^{1,3}_{3,2}\left[\frac{s^{-\sigma}}{a}\middle|\begin{matrix}(\eta-l\pm r,\,\sigma),\ (1-\alpha,\,1)\\(0,\,1),\ (\eta+k-l,\,\sigma)\end{matrix}\right],$$

we arrive at the result after a little simplification.

To justify the inversion of the order of integration in (3.5) we observe that if we take

$$U \equiv t^{-\eta-\frac{1}{2}}\exp\left(-\tfrac{1}{2}tx^{1/\sigma}\right)W_{k+\frac{1}{2},r}(tx^{1/\sigma})\,\varphi(t)$$

$$U\begin{cases}\sim O(t^{-\eta\pm r+\gamma}) & \text{for small } t\\[2ex] \sim O\left(t^{\beta+k-\eta}\exp\left(-t(x^{1/\sigma}-a)\right)\right) & \text{for large } t\,,\end{cases}$$

thus t-integral in (3.5) is absolutely convergent when $R(a)\leqslant 0$ and $R(\gamma-\eta\pm \pm r+1)>0$.

Also if we take

$$U \equiv (s+x)^{-\alpha}x^{1/\sigma(c\sigma-\eta-\sigma+\frac{1}{2})}\exp\left(-\tfrac{1}{2}tx^{1/\sigma}\right)W_{k+\frac{1}{2},r}(tx^{1/\sigma})$$

$$U\begin{cases}\sim O\left(x^{1/\sigma(c\sigma-\eta-\sigma\pm r+1)}\right) & \text{for small } x\\[2ex] \sim O\left(x^{1/\sigma(c\sigma-\eta-\sigma+k+1-\alpha\sigma)}\exp-tx^{1/\sigma}\right) & \text{for large } x\,.\end{cases}$$

Therefore x-integral in (3.5) is absolutely convergent when $R(c\sigma - \eta \pm r + 1) > 0$ and $\sigma > 0$. Also $\psi(s)$ exists when $R(\gamma - \eta \pm r + 1) > 0$, $R(\gamma + \alpha\sigma - c\sigma) > 0, \sigma > 0$, $R(s) > 0$ and $R(c\sigma - \beta) > 0$.

Consequently the inversion of the order of integration in (3.5) is justified by virtue of De La Vallee Poussin's theorem ([**2**] p. 504). This completes the proof of Theorem 2.

Corollary 1. *On replacing s by p and then putting $\sigma = N/S$ in Theorem 2, we get the following result: If*

$$\psi(p) = S\{x^{c+1} f(x); \alpha; p\}$$

and

$$p^{N/S(l+1)} f(p^{N/S}) = W\{\varphi(x); \eta + \tfrac{1}{2}; k + \tfrac{1}{2}, r; p\},$$

then

$$(3.8)\quad \begin{cases} \psi(p) = p^{c+1-\alpha+S/N} S^{\alpha-1} N^{k-\eta-\frac{1}{2}} (2\pi)^{\frac{3}{2}-N/2-S} \times \\ \times \displaystyle\int_0^\infty \varphi(x)\, G^{2N+S,S}_{S+N,2N+S}\left[p^S \left(\frac{x}{N}\right)^N \middle| \begin{matrix} \Delta(S, 1-c-S/N),\ \Delta(N, -k-\eta) \\ \Delta(N, -\eta \pm r),\ \Delta(S, \alpha - c - S/N) \end{matrix}\right] dx, \end{cases}$$

where $\phi(x) \in (\gamma, \beta, a)$ $[R(a) \leqslant 0]$, $R(p) > 0$, $R(cN - \beta S) > 0$, $R(-\eta \pm r + \gamma + 1) > 0$, $R(cN - kS + S \pm rS) > 0$, *and* $R(\gamma S + \alpha N - cN) > 0$.

Corollary 2. *On replacing l by $S\sigma/N - 1$, c by $-Sl$, η by $-r$ and k by $k - \frac{1}{2}$ in Corollary 1, we get a known result* ([**8**] p. 710).

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Abstract.

In this paper first we establish a theorem exhibiting interesting inter-connections exhisting between two generalized Laplace transforms; the kernel of one of these transforms is a modified Bessel function of the second kind while that of the other is product of exponential and Whittaker functions. Next we obtain a number of interesting theorems involving Laplace transform as special cass of our main theorem; now we evaluate certain integrals involving Fox's H-function (which is the one of the most general funcıions) with the help of our theorems. Next we establish a theorem which interconnects a yet another generalized Laplace transform with a generalized Stieltjes transform.

On account of the general nature of the theorems established in this paper, several theorems obtained earliar follow as their special cases.

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