Riv. Mat. Univ. Parma (4) 1 (1975), 135-141

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# Acyclic color functions on graphs. (**)

## Introduction.

Bounds on the point-arboricity of a graph have been derived utilizing degree properties and connectivity properties of the graph and its subgraphs. In this paper it is shown that the line-connectivity can be used to provide a better bound on the point-arboricity of a graph. More specifically, if we let $\sigma(G)$ denote the maximum line-connectivity of any subgraph of $G$, then an upper bound for the point-arboricity of the graph $G$ is half of $\sigma(G)$ plus one. Two proofs are given, the first involves acyclic color functions and is non-constructive, while the second proof derives a constructive procedure for determining the acyclic color function on the graph $G$. In either case, the acyclic color function requires at most one-half of $\sigma(G)$ plus one color values.

Examples are provided to show that the new upper bound on the point-arboricity of a graph is better than those previously proved.

## 1. - Reduced acyclic color functions.

We consider only finite graphs without loops or multiple lines. For the graph $G$, we let $V(G)$ and $E(G)$ respectively denote the point set and the line set of $G$. A subgraph $H$ of the graph $G$ is said to be an *induced subgraph* of $G$ if every line of $G$ which joins two points of $H$ is also a line of $H$. For a subset $S$ of $V(G)$, the subgraph induced by the set $S$ is denoted by $\langle S \rangle$. A subset $C$ of $E(G)$ is called a *cutset* of $G$ if $G - C$ has more components than $G$. If a cutset $C$ has $n$ lines, then it is called an *n-cutset.*

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An *acyclic color function*, or an *acyclic function*, $f$ on a graph $G$ is an assignement of nonnegative integer values to the points of $G$ such that no cycle of $G$ has all of its points assigned the same value. The acyclic color function $a$ on the graph $G$ is called a *reduced acyclic color function*, or a *reduced acyclic function*, if it satisfies the following condition:

$a(v) = \min\{i$: for each cycle $C$ of $G$ containing $v$, the points of $C$ are not all assigned the same value$\}$, for each point $v$ of $G$.

A reduced acyclic function represents an acyclic function which is in some sense locally minimal. To determine a reduced acyclic function on a graph $G$ begin with any acyclic color function $f$ on $G$, if this acyclic color function is not a reduced acyclic function, then the value assigned to some point of $G$ may be reduced and another acyclic color function obtained. Continuing this process one must eventually obtain a reduced acyclic function $a$ and $a$ is said to be obtained from $f$ by *reduction*. However, a reduced acyclic function for a given graph need not be unique. For example, the graph in Figure 1 has two non isomorphic reduced acyclic functions assigned to it.


Figure 1. *A graph with two reduced acyclic functions.*

It follows from the definition that if the degree of a point is at most one, then a reduced acyclic function must assign the value zero to that point. Likewise, if the point $v$ is not on a cycle, then its value must be zero. It is not difficult to see that for any reduced acyclic function $a$ on the graph $G$ and for any point $v$ of $G$,

$$a(v) \leqslant [d(v)/2],$$

where $d(v)$ denotes the degree of $v$ in $G$. We are now ready to prove some results about reduced acyclic functions.

Lemma 1. *Let $S$ be an $n$-cutset of the graph $G$ and let $G_1$ and $G_2$ be disjoint induced subgraphs of $G - S$ such that $G = \langle V(G_1) \cup V(G_2) \rangle$. Let $a_1$ and $a_2$ be reduced acyclic functions on $G_1$ and $G_2$ respectively. Let $m = \max\{a_1(v): v \in V(G_1)\}$. Then there exists a reduced acyclic function $a$ on the graph $G$ such that*

$$a(v) \begin{cases} = a_2(v) & \text{if } v \in V(G_2)\,, \\ \leqslant \max\{m, [n/2]\} & \text{if } v \in V(G_1)\,. \end{cases}$$

Proof. Let $\{H_i\}_{i=0,1,\dots m}$ be an $m+1$ partition of $V(G_1)$ such that $a_1$ is constant over each set $H_i$, $a_1(v) = i$ if $v \in H_i$, and such that each of the sets $H_0, H_1, \dots, H_t$ is incident with at least two lines of $S$ and each of the sets $H_{t+1}, H_{t+2}, \dots, H_m$ is incident with at most one line of $S$. Let $m' = \max\{a_2(v): v \in V(G_2)\}$ and let $\{K_i\}_{i=0,1,\dots,m'}$, be a partition of $V(G_2)$ such that $a_2$ is constant over each $K_i$. Define the function $f$ on $G$ as follows: (1) for each $v \in V(G_2)$, let $f(v) = a_2(v)$; (2) for each $j$, $0 \leqslant j \leqslant m$, let $f(v)$, $v \in H_j$, be the minimum nonnegative integer different from both the one assigned to $H_i$, $0 \leqslant i < j$, and the integer $s$ if there are at least two lines of $S$ joining points of $H_j$ to points of $K_s$.

Since there are at most $[n/2] - t$ pairs of lines of $S$ joining the set $H_j$ to a set $K_i$, the choice of a nonnegative integer for $H_j$ need only be from $j + [n/2] - t$ different nonnegative integers. Hence the maximum nonnegative integer required for the sets $H_0, H_1, \dots, H_t$ is at most $[n/2]$. Furthermore, the nonnegative integer $i$ associated with the set $H_j$, $t < j \leqslant m$, satisfies the inequality $i \leqslant j$, so that $f(v) \leqslant \max\{m, [n/2]\}$ for each $v \in V(G_1)$.

Clearly $f$ is an acyclic color function on $G$. Let $a$ be a reduced acyclic function on $G$ obtained from $f$ by reduction. Since $a_2$ was a reduced acyclic function of $G_2$, it follows that $a(v) = f(v) = a_2(v)$ for each $v \in V(G_2)$. Furthermore, $a(v) \leqslant f(v)$ for each $v \in V(G_1)$, which completes the proof.

## 2 - Bounds on the point-arboricity.

The *point-arboricity* $\varrho(G)$ of the graph $G$ may be characterized as the minimum number of color values required in any acyclic color function on the graph $G$. Clearly $G$ must posses a reduced acyclic function with maximum value $\varrho(G) - 1$. An upper bound on the point-arboricity of a graph is now given in terms of cutsets and the point-arboricity of the resulting subgraphs.

Corollary 1. *Let $S$ be an $n$-cutset of the graph $G$ and let $G_1$ and $G_2$ be disjoint induced subgraphs of $G - S$ such that $G = \langle V(G_1) \cup V(G_2) \rangle$. Then the* *point-arboricity* $\varrho(G)$ *satisfies the inequality*

$$\varrho(G) \leqslant \max\{\varrho(G_1), \varrho(G_2), 1+[n/2]\}. \tag{1}$$

Proof. The graphs $G_1$ and $G_2$ have reduced acyclic functions with maximum values $\varrho(G_1)-1$ and $\varrho(G_2)-1$ respectively. By Lemma 1, $G = \langle V(G_1) \cup \cup V(G_2)\rangle$ has a reduced acyclic function with maximum value at most

$$\max\{\varrho(G_1)-1, \quad \varrho(G_2)-1, \quad [n/2]\},$$

and (1) follows.

We point out here that by a simple modification of the proof of Lemma 1, Corollary 1 may be proved without the use of reduced acyclic functions.

We note that inequality (1) must produce an equality unless the maximum on the right is attained by $1+[n/2]$. We now investigate the case where the $n$-cutset $S$ is a *minimum cutset* of $G$, that is, $S$ has the fewest number of lines possible for any cutset of the graph $G$. In this case, the nonnegative integer $n$ is called the *line-connectivity* of $G$ and is denoted by $\lambda(G)$. Since the trivial graph $K_1$ has no lines, no line-connectivity is associated with it.

A graph $G$ is said to be *critical with respect to point-arboricity*, or simply *critical*, if for each proper subgraph $H$ of $G$, $\varrho(H) < \varrho(G)$. For each positive integer $n \geqslant 2$, the complete graph $K_{2n-1}$ has point-arboricity $n$ and is critical. It is easy to see that the only critical graphs with point-arboricity two are the cycles. The following result provides an upper bound for the point-arboricity of critical graphs in terms of the line-connectivity (see [**3**]).

Lemma 2. (Boucher) *Let $G$ be a critical graph. Then*

$$\varrho(G) \leqslant 1+[\lambda(G)/2]. \tag{2}$$

Proof. Let $S$ be a $\lambda(G)$-cutset of $G$ and let $G_1$ and $G_2$ be the components of $G-S$. From Corollary 1 it follows that

$$\varrho(G) \leqslant \max\{\varrho(G_1), \varrho(G_2), \quad 1+[\lambda(G)/2]\}.$$

Since $G$ is critical, $\varrho(G_1) < \varrho(G)$ and $\varrho(G_2) < \varrho(G)$. Thus (2) follows.

Clearly (2) does not hold for arbitrary graphs, since the disconnected graph $2K_{2n-1}$ has point-arboricity $n$, but line-connectivity zero.

In [**5**] Matula defined the *strength* $\sigma(G)$ of the graph $G$ as follows:

$$\sigma(G) = \max\{\lambda(H) \colon H \text{ is a subgraph of } G\}.$$

Theorem 1. *For any graph G,*

$$\varrho(G) \leqslant 1 + [\sigma(G)/2]\,. \tag{3}$$

Proof. By deleting points and lines from $G$ we must eventually obtain a subgraph $H$ of $G$ that has $\varrho(H) = \varrho(G)$, but which is critical. Then Lemma 2 implies that $\varrho(H) \leqslant 1 + [\lambda(H)/2]$. Since $\lambda(H) \leqslant \sigma(G)$, inequality (3) follows.

From Theorem 1 it is evident that every graph $G$ has a reduced acyclic function which is bounded by $[\sigma(G)/2]$, however the proof of this theorem provides no insight into how to construct such a reduced acyclic function. For applications it is usually useful to actually have a constructive approach. We now demonstrate a method for constructing a reduced acyclic function on the graph $G$ bounded by $[\sigma(G)/2]$.

In order to provide a constructive procedure, it is convenient to utilize the concept of a «slicing» of a graph (see [**4**]). For notational purposes, let $C_0 = \phi$. The ordered partition of the lines of the graph $G$, $Z = (C_1, C_2, \ldots, C_m)$, is a *slicing* of $G$ if $C_i$ is a cutset of

$$G - \bigcup_{j=0}^{i-1} C_j \qquad (1 \leqslant i \leqslant m)\,. \tag{4}$$

Furthermore, $Z$ is called a *narrow slicing* of $G$ if each cutset $C_i$, $1 \leqslant i \leqslant m$, is a minimum cutset of some component of (4). We list a result of MATULA [**5**] which is necessary for the following construction.

(A) *The maximum number of lines in any cutset of a narrow slicing equals the strength of the graph.*

It was shown in [**5**] that a narrow slicing of a graph can be found constructively from the slicing algorithm. We now utilize Lemma 1 and (A) to construct a reduced acyclic function on $G$ bounded by $[\sigma(G)/2]$.

Assume that $G$ is not a totally disconnected graph, for then $a(v) = 0$ for each $v \in V(G)$ is the required reduced acyclic function. Let $Z = (C_1, C_2, \ldots, C_m)$ be a narrow slicing of $G$. Then $G - \bigcup_{i=0}^{m} C_i$ is a totally disconnected graph and we define the reduced acyclic function $a_m$ such that $a_m(v) = 0$ for each $v \in V(G)$. The constructive procedure of Lemma 1 can be utilized to provide a reduced acyclic function $a_{m-1}$ on $G - \bigcup_{i=0}^{m-1} C_i$ bounded by $[\sigma(G)/2]$, since (A) implies that $|C_m| \leqslant \sigma(G)$. Proceeding recursively, if $a_j$ is a reduced acyclic function on

$G - \bigcup_{i=0}^{j} C_i$ bounded by $[\sigma(G)/2]$, then Lemma 1 may be utilized to yield a reduced acyclic function $a_{j-1}$ on $G - \bigcup_{i=0}^{j-1} C_i$ bounded by $[\sigma(G)/2]$, since (A) implies that $|C_j| \leq \sigma(G)$. This procedure ends with a reduced acyclic function $a$ on $G$ bounded by $[\sigma(G)/2]$. We have thus produced a constructive proof of Theorem 1.

Theorem 1 provides a stronger upper bound on the point-arboricity than either of the following corollaries proved in [1] and [2]. Let $\Delta(G)$ and $\delta(G)$ denote respectively the maximum and minimum degree of $C$.

Corollary 1a. (CHARTRAND, KRONK, and WALL) *For any graph $G$,*

$$\varrho(G) \leq 1 + [\Delta(G)/2]. \tag{5}$$

Corollary 1b. (CHARTRAND and KRONK) *For any graph $G$,*

$$\varrho(G) \leq 1 + [(\max \delta(H))/2], \tag{6}$$

*where the maximum is taken over all subgraphs $H$ of $G$.*

The first corollary follows from the inequality $\sigma(G) \leq \Delta(G)$. From the inequality $\lambda(H) \leq \delta(H)$ for any subgraph $H$ of $G$, it follows that

$$\sigma(G) \leq \max\{\delta(H) \colon H \text{ is a subgraph of } G\}.$$

The results (5) and (6) are in general weaker than (3), as can be seen from the graph $G$ in Figure 2. Here $\delta(G) = \Delta(G) = 4$, while $\sigma(G) = 3$.

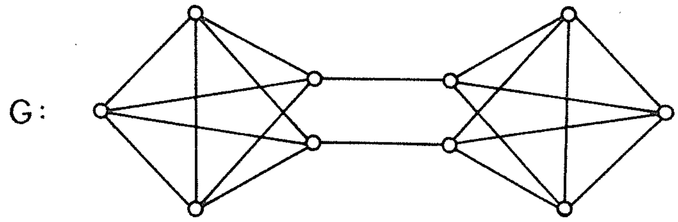


Figure 2. *A graph $G$ with $\delta(G) = \Delta(G) = 4$ and $\sigma(G) = 3$.*

**References.**

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## Abstract.

*The point-arboricity of a graph G is the minimum number of color classes needed to color the points of G so that each color class induces an acyclic subgraph of G. Acyclic color functions are defined and a discussion of how to construct such functions is provided. These functions are used to give a new bound on the point-arboricity in terms of the maximum line-connectivity of any of its subgraphs.*

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