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A sufficient condition for the existence of an hamiltonian chain in a finite simple graph. (**)

1. – We will consider finite and simple graphs only.

Let's start by giving some definitions and notations ([1], [5]).

A *clique* of a graph G is a maximal complete subgraph of G. The *clique graph* $K(G)$ of $G = (V, E)$ is the graph $K(G) = (V', E')$, whose set of vertices V' is in one to one correspondence with the set of distinct cliques of G, and two vertices u', v' are adjacent in $K(G)$, i.e. $\{u', v'\} \in E'$, if and only if the two cliques of G to which these two vertices correspond have a vertex in common.

We will conform to the following convention. Let an upper case letter C, affected if necessary by subscripts or superscripts, indicate a clique of G, then the same lower case letter, with the same subscripts and superscripts denotes the vertex of $K(G)$ corresponding to this clique in the aforementioned correspondence.

$|G|$ denotes the order of the graph G. $E(G)$, $V(G)$ denote the set of edges and the set of vertices of G, respectively.

$E_C(x)$ is the set of the edges of C, which meet at vertex x; $C-(x)$ is the clique of order $|C|-1$, defined by $V(C-(x)) = V(C) - \{x\}$, $E(C-(x)) = E(C) - E_C(x)$.

If G, G' are isomorphic, then we write $G = G'$.

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Let $\gamma = (x_1, x_2, \ldots, x_p)$ be an elementary chain of G. *γ-neighbour of x_i* is each of the two vertices x_{i-1}, x_{i+1}, if $i = 2, \ldots, p-1$; $x_2(x_p)$ is the γ-neighbour of $x_1(x_{p-1})$.

2. – Let G be a graph, such that no two distinct cliques C_i, C_j belonging to it have more than one vertex in common, i.e. $|C_i \cap C_j| \leqslant 1$, and let $K(G)$ be triangle-less and of order n, then the following properties hold:

P1 [**2**]. The set $\{E(C_1), \ldots, E(C_n)\}$ of parts of $E(G)$ is a partition of $E(G)$.
Indeed the hypothesis $|C_i \cap C_j| \leqslant 1$, for any two cliques of G, implies that each edge of G lies in exactly one clique.

P2 [**2**]. No vertex of G lies in more than two cliques.
Indeed if a vertex were in three cliques C, C', C'' of G, in $K(G)$ there would exist the triangle (c, c', c''), which contradicts the hypothesis that $K(G)$ has no triangles.

P3. If the vertex c has degree d in $K(G)$, then $|C| \geqslant d$.
Indeed, let $c_1, c_2, \ldots, c_d$ be the vertices of $K(G)$, neighbours of c. Let $\{x_j\} = V(C) \cap V(C_j)$, $j = 1, \ldots, d$. Property P2 implies that there is no clique of G, distinct from C and C_j having x_j as a vertex. Therefore two distinct cliques, adjacent to C, contain two distinct vertices of C. Therefore $|C| \geqslant d$.

3. – We are now able to prove the following

Theorem. *If $|C_i \cap C_j| \leqslant 1$ for any two distinct cliques C_i, C_j of G and if $K(G)$ is triangle-less and has an H-chain, then G has an H-chain.*

Proof. The result is clearly true when $\max_{1 \leqslant i \leqslant n} |C_i| = 2$. We just observe that by the hypotheses $|C_i| = 2$, $i = 1, \ldots, n$, and G is connected. Furthermore, for any vertex x of G, we have that $d_G(x) \leqslant 2$, because if x were in three distinct edges, i.e. three cliques C_i, C_j, C_k, then $K(G)$ would have the triangle (c_i, c_j, c_k).

We proceed by induction on the highest order k of the cliques: we suppose that any graph whose cliques have order $\leqslant k-1$, $k \geqslant 3$, has an H-chain, and we show that there exists an H-chain in any graph each of whose cliques is of order $\leqslant k$.

Let $C_{i_1}, \ldots, C_{i_l}$ be the cliques of G, of order k, and γ an H-chain of $K(G)$. Let c', c'' be the vertices of $K(G)$ γ-neighbours of vertex c_{i_s}, $s = 1, \ldots, l$ (1).

(1) If there is only one vertex γ-neighbour of c_{i_s}, the proof is exentially the same.

We set

$$\{x'\} = V(C_{i_s}) \cap V(C'_s), \qquad \{x''\} = V(C_{i_s}) \cap V(C''_s).$$

For a given clique C_{i_s}, by property P2 either

(i) every vertex of C_{i_s}, distinct from x', x'', belongs to no other clique of G, or

(ii) there is a vertex x_s of C_{i_s}, distinct from x', x'', and there is exactly one clique C, distinct from C_{i_s}, such that $\{x_s\} = V(C_{i_s}) \cap V(C)$.

In the first case, if x_s denotes a vertex of C_{i_s}, distinct from x', x'', we construct the graph

$$\big(V(G) - \{x_s\},\, E(G) - E_{C_{i_s}}(x_s)\big).$$

In the second case we construct the graph

$$\big(V(G),\, E(G) - E_{C_{i_s}}(x_s)\big).$$

Let $G^{(1)}$ denote the graph obtained in either cases.

By P1, passing from G to $G^{(1)}$, the orders of the cliques distinct from C_{i_s} and the adjacency relation between the cliques are preserved, except for the pair (C, C_{i_s}).

In case (i) $K(G) = K(G^{(1)})$. In case (ii) $K(G^{(1)})$ is the spanning subgraph of $K(G)$ obtained by suppressing the edge $\{c_{i_s}, c\}$.

In either case γ is an H-chain of $K(G^{(1)})$, because the edge $\{c_{i_s}, c\}$ is not on γ.

If $l = 1$, then the highest order of the cliques of $G^{(1)}$ is $k-1$ and by the induction hypothesis there exists an H-chain $\Gamma^{(1)}$ in $G^{(1)}$.

In case (i), let h' be an H-chain of the clique $C_{i_s} - (x_s)$, having end-vertices x', x''. Any H-chain of $G^{(1)}$ clearly has a sub-chain of the same type as h'. Therefore

$$\Gamma^{(1)} = \Gamma_1 h' \Gamma_2,$$

Γ_1, Γ_2 being suitable chains of $G^{(1)}$.

Consider now the chain

$$\Gamma = \Gamma_1 h \Gamma_2,$$

h being an H-chain of C_{i_s}, having end-vertices x', x''; Γ is evidently an H-chain of G.

In case (ii) the H-chain $\Gamma^{(1)}$ of $G^{(1)}$ is an H-chain of G too.

Therefore, in both cases there is in G an H-chain.

If $l\geqslant 2$, let $G^{(0)}=G$, $G^{(1)},\ldots,G^{(l)}$, be a finite sequence of graphs such that $G^{(i+1)}$ is obtained from $G^{(i)}$ by substituting a clique of order k with one of order $k-1$, with the same procedure we followed passing from G to $G^{(1)}$.

The highest order of the cliques of $G^{(0)},\ldots,G^{(l-1)}$ is k, the highest order of the cliques of $G^{(l)}$ is $k-1$.

γ is an H-chain of $K(G^{(l)})$, therefore $G^{(l)}$ has an H-chain $\Gamma^{(l)}$.

In order to find an H-chain of G, from $\Gamma^{(l)}$ we construct an H-chain of graph $G^{(l-1)}$ by the aforementioned procedure and by iterating l times an H-chain Γ of G from $\Gamma^{(l)}$. Q.E.D.

It is possible to show by the same method that if any two distinct cliques C_i, C_j of G satisfy the condition $|C_i\cap C_j|\leqslant 1$, if $K(G)$ is triangle-less, and has an H-cycle, then G is an H-graph.

4. – In the hypotheses of the theorem, given an H-chain of $K(G)$ we can provide an algorithm for finding an H-chain of G.

Let $\gamma=(c_1,\ldots,c_n)$ be an H-chain of $K(G)$ and let x_i be the vertex common to the cliques C_i, C_{i+1}, $i=1,\ldots,n-1$; let x be a vertex of C_1, distinct from x_1; let x_n be a vertex of C_n, distinct from x_{n-1}. There are two cases to be considered:

(i) Every vertex of $K(G)$ has degree $\leqslant 2$.

The vertices $c_2,\ldots,c_{n-1}$ clearly have degree 2 and c_1, c_n have degree 1; every clique C_i of G, $i=2,\ldots,n-1$, is adjacent to C_{i-1}, C_{i+1}; C_1 (C_n) is adjacent to C_2 (C_{n-1}). By P2 $x_{i-1}\neq x_i$, $i=1,\ldots,n$.

Therefore, if we call γ_i an H-chain of C_i, having end-vertices x_{i-1}, x_i, the chain $\Gamma=\gamma_1\gamma_2\ldots\gamma_n$ is an H-chain of G.

(ii) There is a vertex of $K(G)$ of degree $\geqslant 3$.

Let $c_{j_1},\ldots,c_{j_r}$, $j_1<\ldots<j_r$, $r\geqslant 1$, be the vertices of $K(G)$ of degree $\geqslant 3$.

Call $c_{11},\ldots,c_{1q}$, $q_1\geqslant 1$, the vertices of $K(G)$ neighbours of c_{j_1}, but not γ-neighbours of c_{j_1}.

We set

$$\{x_{1t}\}=V(C_{j_1})\cap V(C_{1t})\,,\qquad t=1,\ldots,q_1\,.$$

Let G_1 be the graph $(V(G),E(G)-\bigcup_t E_{C_{1t}}(x_{1t}))$.

If $|C_{1t}|>2$, for any $t=1, \ldots, q_1$, it is easy to see that:

$$K(G_1)=V(K(G)) , \qquad E(K(G))-(\{c_{j_1}, c_{11}\}, \{c_{j_1}, c_{12}\}, \ldots, \{c_{j_1}, c_{1q_1}\}) .$$

Indeed, in the passage from G to G_1 the cliques $C_{11}, \ldots, C_{1q}$ turn, by P1, into cliques of order $|C_{1t}|-1\geqslant 2$, and the remaining cliques are preserved.

Therefore $V(K(G))=V(K(G_1))$.

We now prove that

$$E(K(G_1))=E(K(G))-(\{c_{j_1}, c_{11}\}, \ldots, \{c_{j_1}, c_{1q}\}) .$$

All the other cases being trivial, we just need to prove that all edges of $E(K(G))$ distinct from $\{c_{j_1}, c_{1t}\}$ and of the form $\{c_i, c_{1t}\}$ are in $E(K(G_1))$.

Indeed, if the edge $\{c_i, c_{1t}\}$ were not in $E(K(G_1))$, in $K(G)$ there would be the triangle (c_i, c_{1t}, c_{j_1}), contrary to the hypotheses.

From what we have just said γ is an H-chain of $K(G_1)$.

If there is one (and only one) clique C_{1t} of order 2, then $c_{1t}=c_n$.

Also in this case $K(G_1)$ has an H-chain: $(c_1, \ldots, c_{n-1})$.

If in $K(G_1)$ there is no vertex of degree $\geqslant 3$, then $K(G_1)$, having an H-chain, is of type considered in case (i), and therefore, G_1 has an H-chain, which is evidently an H-chain of G.

Otherwise, calling c_{j_p} the first of vertices $c_{j_2}, \ldots, c_{j_r}$ having degree $\geqslant 3$, we construct the graph

$$G_2=(V(G), E(G_1)-\bigcup_a E_{c_{pa}}(x_{pa})) ,$$

where c_{pa}, $a=1, \ldots, q_p$, denotes a vertex of $K(G_1)$ neighbour of c_{j_p} but not γ-neighbour of c_{j_p}.

By this procedure, we end up by constructing a sequence $G_0=G, G_1, G_2, \ldots$ of graphs such that G_i is a spanning subgraph og G_{i-1}, $i\geqslant 1$, and every clique of graph $K(G_i)$ has an H-chain.

The procedure ends after at most r steps, with the construction of a graph G_h, such that every vertex of $K(G_h)$ has degree $\leqslant 2$.

By the result of case (i), G_h has an H-chain which is also an H-chain of G.

As a concluding remark, we notice that, by similar procedure, we can construct an H-cycle of G if γ is an H-cycle of $K(G)$.

References.

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Sommario.

Se due qualunque cliques di un grafo G finito semplice hanno al più un vertice in comune, se il clique-grafo di G è privo di triangoli e ha una catena hamiltoniana, allora G ha una catena hamiltoniana.

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