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Stieltjes transformable generalized functions. (**)

1. - Introduction.

In a recent paper PANDEY [1] has given a generalization of classical STIELTJES transform to certain classes of generalized functions. He introduced the space $S'_\alpha(I)$ of generalized functions, where α is a fixed real (arbitrary) constant less than or equal to 1, and showed that the real and complex inversion formulae of WIDDER ([3], pp. 125, 144) for STIELTJES transforms are still valid when the limiting operation in those formulae is understood as weak convergence in the space $\mathscr{D}'$ of SCHWARTZ distributions. The testing function space which was dealt within [1] was defined over $\mathscr{R}^1_+$. In this Note we shall deal with an S_α-space defined over $\mathscr{R}^n_+$, the n-dimensional euclidean space of non-negative real numbers, and α will be fixed arbitrary element of $\mathscr{R}^1$. Our object is to find a representation formula for a certain subspace of $S'_\alpha(\mathscr{R}^n_+)$.

The notation and terminology will follow that of [1] and [4]. Unless otherwise stated x will be understood to be a variable in $\mathscr{R}^n_+$ and α will signify a constant in $\mathscr{R}^1$. If a and b are in $\mathscr{R}^n_+$, by $a > b$ we mean that $a_i > b_i$ for $i = 1, 2, 3, \ldots, n$, where a_i and b_i are components of a and b respectively. When c and x both belong to $\mathscr{R}^n_+$ the expression cx is understood to be scalar product of c and x.

2. - The testing function space $S_\alpha(\mathscr{R}^n_+)$.

Let $\mathscr{R}^n_+$ stand for the n-dimensional euclidean space of non-negative real numbers, and let $x = \{x_1, x_2, \ldots, x_n\} \in \mathscr{R}^n_+$. We introduce the following nota-

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tions:

$$|x| \triangleq x_1 + x_2 + \dots + x_n\,, \qquad x^i \triangleq x_1^{i_1}\, x_2^{i_2} \dots x_n^{i_n}\,,$$

$$\left(\frac{\partial}{\partial x}\right)^i \triangleq \left(\frac{\partial}{\partial x_1}\right)^{i_1}\left(\frac{\partial}{\partial x_2}\right)^{i_2} \cdots \left(\frac{\partial}{\partial x_n}\right)^{i_n},$$

$$\mathscr{D}^k\varphi = \left(x\frac{\partial}{\partial x}\right)^k \varphi \triangleq \left(x_1\frac{\partial}{\partial x_1}\right)^{k_1}\left(x_2\frac{\partial}{\partial x_2}\right)^{k_2} \cdots \left(x_n\frac{\partial}{\partial x_n}\right)^{k_n} \varphi(x_1, x_2, \dots, x_n)\,,$$

where $k = (k_1, k_2, \dots, k_n)$ and the k_i are non-negative integers. The order of the differentiation operator $\mathscr{D}^k$ will be defined as the number

$$|k| = k_1 + k_2 + \dots + k_n\,.$$

A complex valued and infinitely differentiable function $\varphi(x)$ defined over $\mathscr{R}^n_+$ is said to belong to the space $S_\alpha(\mathscr{R}^n_+)$ if

$$\gamma_k(\varphi) = \max_{|k|\leq m} \sup_{x\in\mathscr{R}^n_+} (1 + |x|)^\alpha \left|\left(x\frac{\partial}{\partial x}\right)^k \varphi(x)\right| < \infty\,, \tag{1}$$

for $m = 0, 1, 2, \dots$ and for a fixed real number $\alpha \in \mathscr{R}^1$ less than or equal to 1. Clearly $S_\alpha(\mathscr{R}^n_+)$ is a vector space. The space $\mathscr{D}(\mathscr{R}^n_+)$ is a subspace of $S_\alpha(\mathscr{R}^n_+)$ and the topology of $\mathscr{D}(\mathscr{R}^n_+)$ is stronger than the topology induced on $\mathscr{D}(\mathscr{R}^n_+)$ by $S_\alpha(\mathscr{R}^n_+)$ and as such restriction of any member of $S'_\alpha(\mathscr{R}^n_+)$ to $\mathscr{D}(\mathscr{R}^n_+)$ is in $\mathscr{D}'(\mathscr{R}^n_+)$. In case $n = 1$, we write $\mathscr{R}^1_+ = I = (0, \infty)$ and $S_\alpha(\mathscr{R}^1_+)$ becomes PANDEY's $S_\alpha(I)$ space.

The convergence in $S_\alpha(\mathscr{R}^n_+)$.

A sequence $\{\varphi_\nu(x)\}_{\nu=1}^\infty$, where $\varphi_\nu(x)$ is in $S_\alpha(\mathscr{R}^n_+)$ for each ν, is said to converge to $\varphi(x)$ in $S_\alpha(\mathscr{R}^n_+)$ if $\gamma_k(\varphi_\nu - \varphi) \to 0$ as $\nu \to \infty$ for each $k = 0, 1, 2, \dots$. We further add that a sequence $\{\varphi_\nu(x)\}_{\nu=1}^\infty$ where each $\varphi_\nu(x) \in S_\alpha(\mathscr{R}^n_+)$, is a CAUCHY sequence in $S_\alpha(\mathscr{R}^n_+)$ if $\gamma_k(\varphi_\nu - \varphi_\mu) \to 0$ as μ and ν both go to infinity, independently of each other, for $k = 0, 1, 2, \dots$. It has been proved by PANDEY [1] that for $n = 1$, $S_\alpha(\mathscr{R}^n_+)$ is a locally convex HAUSDORFF topological vector space. It can similarly be proved that the result is also true for $S_\alpha(\mathscr{R}^n_+)$ for $n > 1$.

The dual space $S'_\alpha(\mathscr{R}^n_+)$ contains all distributions of compact support in $\mathscr{R}^n_+$;

3. - Representation.

Now we shall prove a representation theorem for STIELTJES transformable generalized functions. Our proof is analogous to the method employed in structure theorem for SCHWARTZ distributions ([2], pp. 272-274).

Theorem. *Let f be an arbitrary element of $S'_\alpha(\mathscr{R}^n_+)$ and φ be an element of $\mathscr{D}(\mathscr{R}^n_+)$, the space of infinitely differentiable functions with compact support in $\mathscr{R}^n_+$. Then, there exist N continuous functions $h_i(x)$ defined over $\mathscr{R}^n_+$ such that*

$$\langle f, \varphi\rangle = \langle \sum_{|i|\leqslant r+n} (-1)^{|i|} (\partial/\partial x)^i [(1+|x|)^{\alpha-n} x^{i-1} P_i(x) \rhd h_i], \varphi(x)\rangle . \tag{2}$$

Here $\alpha \in \mathscr{R}^1$ is a fixed real number always less than or equal to 1, N is the number of n-tuples i satisfying $|i|\leqslant n+r$, r is an appropriate non-negative integer and $\rhd$ is the differentiation monomial $\partial/\partial x_1, \partial/\partial x_2, \ldots, \partial/\partial x_n$ and $P_i(x)$ are polynomial of degree $n+1$.

Proof. Let $\{\gamma_k\}_{k=0}^\infty$ be the sequence of seminorms as defined in section 3 and let f and φ be arbitrary elements of $S'_\alpha(\mathscr{R}^n_+)$ and $\mathscr{D}(\mathscr{R}^n_+)$ respectively. Then by the boundedness property of generalized functions, we have for an appropriate constant C and a non-negative integer r,

$$\left\{\begin{aligned} \langle f, \varphi\rangle &\leqslant C \max_{|k|\leqslant r} \gamma_k(\varphi) ,\\ &\leqslant C \max_{|k|\leqslant r} \sup_{x\in\mathscr{R}^n_+} (1+|x|)^\alpha \left|\left(x\frac{\partial}{\partial x}\right)^k \varphi(x)\right| ,\\ &\leqslant C \max_{|k|\leqslant r} \sup_{x} (1+|x|)^\alpha \prod_{j=1}^n \left|\sum_{i_j=0}^{k_j} (a_i)_{i_j} (x_j)^{i_j} \left(\frac{\partial}{\partial x_j}\right)^{i_j} \varphi\right| ,\\ &\leqslant C' \max_{|k|\leqslant r} \sup_{x} (1+|x|)^\alpha \prod_{j=1}^n k_j \max_{0\leqslant i_j\leqslant k_j} (x_j)^{i_j} \left(\frac{\partial}{\partial x_j}\right)^{j} \varphi ,\end{aligned}\right. \tag{3}$$

where

$$C' = C \prod_{j=1}^n \max_{0\leqslant i_j\leqslant k_j} |(a_j)_{i_j}| .$$

So that

$$\langle f, \varphi\rangle \leqslant C'' \max_{|i|\leqslant r} \sup_{x} (1+|x|)^\alpha x^i \left|\left(\frac{\partial}{\partial x}\right)^i \varphi\right| \tag{4}$$

where $C'' = C' \prod_{j=1}^n k_j$.

For any $\Psi \in \mathscr{D}(\mathscr{R}^n_+)$ we can write

$$\sup_x |\Psi(x)| \leqslant \sup_x \left| \int_{x_1}^{\infty} \dots \int_{x_n}^{\infty} \rhd \psi(x)\, dx_1 \dots dx_n \right| \leqslant \| \rhd \psi \|_{L^1}, \tag{5}$$

where $\rhd$ is the differentiation monomial $\partial/\partial x_1, \partial/\partial x_2, \dots, \partial/\partial x_n$. Therefore from (4), we have

$$\begin{cases} |\langle f, \varphi\rangle| \leqslant C'' \max\limits_{|i|\leqslant r} \left\| \rhd (1+|x|)^\alpha x^i \left(\dfrac{\partial}{\partial x}\right)^i \varphi(x) \right\|_{L'}, \\ \qquad \leqslant C'' \max\limits_{|i|\leqslant r+n} \left\| (1+|x|)^{\alpha-n} x^{i-1} P_i(x) \left(\dfrac{\partial}{\partial x}\right)^i \varphi(x) \right\|_{L'}, \end{cases} \tag{6}$$

where $P_i(x)$ are polynomials of degree $n+1$.

Let the number of n-tuples i satisfying $|i| \leqslant \gamma + n$ be denoted by N and the product space $L' \times \overset{N}{\dots} \times L'$ by $(L')^N$. We consider the linear one-to-one mapping

$$\tau : \varphi \to \left\{ (1+|x|)^{\alpha-n} x^{i-1} P_i(x) \left(\frac{\partial}{\partial x}\right)^i \varphi(x) \right\}_{|i|\leqslant r+n} \tag{7}$$

of $\mathscr{D}(\mathscr{R}^n_+)$ into $(L')^N$. In view of (6) we see that the linear functional $\tau\varphi_r \to \langle f, \varphi \rangle$ is continuous on $\tau\mathscr{D}(\mathscr{R}^n_+)$ for the topology induced by $(L')^N$. Hence by HAHN-BANACH theorem, it can be extended as a continuous linear functional in the whole of $(L')^N$. But the dual of $(L')^N$ is isomorphic with $(L^\infty)^N$ ([**2**], pp. 214, 259), therefore there exist NL^∞ functions g_i ($|i| \leqslant r+n$) such that $\langle f, \varphi \rangle = \sum\limits_{|i|\leqslant r+n} \langle g_i (1+|x|)^{\alpha-n} x^{i-1} P_i(x) (\partial/\partial x)^i \varphi(x) \rangle$.

So that $\langle f, \varphi \rangle = \sum\limits_{|i|\leqslant r+n} \langle (-1)^{|i|} (\partial/\partial x)^i [(1+|x|)^{\alpha-n} x^{i-1} P_i(x) g_i], \varphi(x) \rangle$.

Therefore $f = \sum\limits_{|i|\leqslant r+n} \langle (-1)^{|i|} (\partial/\partial x)^i [(1+|x|)^{\alpha-n} x^{i-1} P_i(x) g_i], \varphi(x) \rangle$.

For each i we set $h_i(x) = \int_0^{x_1} \dots \int_0^{x_n} g_i(y_1, \dots, y_n)\, dy_1 \dots dy_n$.

Since g_i is in L^∞, we see that h_i is continuous in $\mathscr{R}^n_+$ and that $|h_i(x)| \leqslant |x_1| \dots |x_n| \, \|g_i\|_L^\infty$.

Furthermore, we have $g_i = \rhd h_i$ and consequently

$$f = \sum_{|i|\leqslant r+n} (-1)^{|i|} \left(\frac{\partial}{\partial x}\right)^i [(1+|x|)^{\alpha-n} x^{i-1} P_i(x) \rhd h_i]. \tag{8}$$

This completes the proof.

Taking $n = 1$ in (2) we obtain structure formula for elements of PANDEY's space $S'_\alpha(I)$ in the form

$$\langle f, \varphi\rangle = \langle \sum_{i=0}^{\nu+1} (-1)^i \left(\frac{\partial}{\partial x}\right)^i \left[(1+x)^{\alpha-1} x^{i-1} P_i(x) \frac{\partial}{\partial x} h_i\right] \varphi(x)\rangle$$

where $h_i(x)$ are continuous functions defined over $\mathscr{R}^n_+ = (0, \infty)$.

We shall now define the testing function spaces $\mathscr{S}_{c,d}$ and $\mathscr{S}_{c,d}$, and state some properties of these spaces which can be established by the standard techniques followed by PANDEY [1].

The testing function space $\mathscr{S}_{c,d}$.

Let $c, d \in \mathscr{R}'$ and $s \in \mathscr{C}'$. Let $\xi_{c,d}(x)$ be the function

$$\xi_{c,d}(x) = \begin{cases} x^c & 0 < x < 1\,, \\ x^d & 1 \leqslant x < \infty\,. \end{cases} \tag{9}$$

$\mathscr{S}_{c,d}$ denotes the space of all complex-valued functions $\varphi(x)$ on $I = (0 < x < \infty)$ on which the functionals β_k defined by

$$\beta_k(\varphi) \triangleq \beta_{k,c,d}(\varphi) \triangleq \left|\xi_{c,d}(x) \left(x \frac{\partial}{\partial x}\right)^k \varphi(x)\right|, \qquad k = 0, 1, 2, \ldots, \tag{10}$$

assume finite values. The countable set of seminorms $\{\beta_k\}_{k=0}^\infty$ generates the topology for $\mathscr{S}_{c,d}$. It can be shown that $\mathscr{S}_{c,d}$ is HAUSDORFF, locally convex, first countable, complete, countably normed space. The space $\mathscr{D}(I)$ is a subspace of $\mathscr{S}_{c,d}$ and the topology of $\mathscr{D}(I)$ is stronger than the topology induced on $\mathscr{D}(I)$ by $\mathscr{S}_{c,d}$ and as such the restriction of any member of $\mathscr{S}'_{c,d}$ (the dual space of $\mathscr{S}_{c,d}$) to $\mathscr{D}(I)$ is in $\mathscr{D}'(I)$. We say that a sequence $\{\varphi_\nu(x)\}_{\nu=1}^\infty$ where each $\varphi_\nu(x)$ belongs to $\mathscr{S}_{c,d}$, is a CAUCHY sequence in $\mathscr{S}_{c,d}$ if $\beta_k(\varphi_\mu - \varphi_\nu)$ tends to zero for any non-negative integer k as μ and ν both tend to infinity independently of each other. It can be readily seen that $\mathscr{S}_{c,d}$ is sequentially complete.

For complex s not lying on the negative real axis and $k = 1, 2, 3, \ldots$, $1/(s+x)^k \in \mathscr{S}_{c,d}$.

The testing function space $\underline{\mathscr{S}}_{c,d}$.

An infinitely differentiable complex valued function $\varphi(x)$ defined over I is said to belong to ${}_c\underline{\mathscr{S}}_{,d}$ if

$$\tau_k(\varphi) \triangleq \tau_{k,c,d}(\varphi) \triangleq \left|\xi_{c,d}(x)\, x^k \left(\frac{\partial}{\partial x}\right)^k \varphi(x)\right| < \infty \tag{11}$$

for all $k = 0, 1, 2, \ldots$, where $\xi_{c,d}(x)$ is the same as defined in (9). The concept of convergence and completeness in $\underline{\mathscr{S}}_{c,d}$ is defined in a way similar to those defined in $\mathscr{S}_{c,d}$. The space $\underline{\mathscr{S}}_{c,d}$ is also a locally convex HAUSDORFF topological vector space. The restriction of any member of $\underline{\mathscr{S}}'_{c,d}$ (the dual space of $\underline{\mathscr{S}}_{c,d}$) to $\mathscr{D}(I)$ is in $\mathscr{D}'(I)$.

Following results can be established by following the technique of PANDEY ([**1**], Lemma 1).

(i) The spaces $\mathscr{S}_{c,d}$ and $\underline{\mathscr{S}}_{c,d}$, for fixed real numbers $c \geqslant 0$ and $d \leqslant 1$, are equal in store of elements.

(ii) T_1 the topology generated in $\mathscr{S}_{c,d}$ by the sequence of seminorms $\{\beta_k\}_{k=1}^{\infty}$ is the same as T_2, the topology generated on $\mathscr{S}_{c,d}$ by the sequence of seminorms $\{\tau_k\}_{k=1}^{\infty}$.

4. - The Stieltjes transform on $\mathscr{S}'_{c,d}$.

The STIELTJES transform $F(s)$ of an arbitrary element $f(x)$ of $\mathscr{S}'_{c,d}$ is defined by

$$\mathscr{S}[f] \triangleq F(s) \triangleq \left\langle f(x), \frac{1}{s+x} \right\rangle, \tag{12}$$

for all s lying in the compact set Ω_f of the complex plane not meeting the negative real axis.

Now we are stating some theorems whose proofs are similar to those of PANDEY [**1**] and hence are omitted.

THEOREM 2. (The analiticity theorem). *Let $F(s)$ be the Stieltjes transform of $f(x) \in \mathscr{S}'_{c,d}$ as defined in* (12). *Then, $F(s)$ is analytic on Ω_f and for* $k = 1, 2, 3, \ldots$,

$$F^{(k)}(s) = \left\langle f(x), \frac{(-1)^k k!}{(s+x)^{k+1}} \right\rangle . \tag{13}$$

Moreover, for positive real x,

$$F^{(k)}(x) = \begin{cases} 0(x^{-k}) & \text{as } x \to \infty \text{ if } c \geqslant 0 \text{ and } d \leqslant 1, \\ 0(x^{-k}) & \text{as } x \to \infty \text{ if } c \geqslant 0 \text{ and } d = 1, \\ 0(x^{-k-1}) & \text{as } x \to 0+ \text{ if } c = 0 \text{ and } d \leqslant 1, \\ 0(x^{-k-1}) & \text{as } x \to 0+ \text{ if } c > 0 \text{ and } d \leqslant 1. \end{cases} \tag{14}$$

The last order relation could not be obtained in the Pandey's space $S'_\alpha(I)$.

Theorem 3. (The real inversion formula). *For fixed* $c \geqslant 0$, $d \leqslant 1$ *and* $x > 0$ *let* $F(x)$ *be the Stieltjes transform of* $f(x)$ *belonging to* $\mathscr{S}'_{c,d}$ *defined by* (12). *Then for an arbitrary element* $\varphi(x)$ *of* $\mathscr{D}(I)$ *we have*

$$\langle L_{k,x} F(x),\ \varphi(x)\rangle \to \langle f, \varphi\rangle \qquad as \quad k \to \infty, \tag{15}$$

where

$$L_{k,x}[\psi(x)] = \frac{(-x)^{k-1}}{k!(k-2)!}\frac{\partial^{2k-1}}{\partial x^{2k-1}}[x^k \psi(x)], \tag{16}$$

where $\psi(x)$ *is an element of* $\mathscr{S}'_{c,d}(I)$ *and the differentiation in* (16) *is supposed to be in the distributional sense.*

Theorem 4. (The complex inversion formula). *Let* $f(t)$ *be an arbitrary element of* $\mathscr{S}'_{c,d}$ *and* $F(s)$ *be the Stieltjes transform of* $f(t)$. *Then for an arbitrary element* $\varphi(x) \in \mathscr{D}(I)$ *we have*

$$\left\langle \frac{F(-\xi - i\eta) - F(-\xi + i\eta)}{2\pi i},\ \varphi(\xi)\right\rangle \to \langle f, \varphi\rangle \qquad as\ \eta \to 0+,$$

where $c \geqslant 0$ *and* $d \leqslant 1$.

Theorem 5. (The uniqueness theorem). *If* $\mathscr{S}[f] = F(s)$ *for* $s \in \Omega_f$ *and* $\mathscr{S}[h] = H(s)$ *for* $\mathscr{S} \in \Omega_h$, *if* $\Omega_f \cap \Omega_h$ *is not empty, and if* $F(s) = H(s)$ *for* $\mathscr{S} \in \Omega_f \cap \Omega_h$, *then* $f = h$ *in the sense of equality in* $\mathscr{D}'(I)$.

Theorem 6. (The representation theorem). *Let* f *be an arbitrary element of* $\mathscr{S}'_{c,d}$ *and* φ *be an element of* $\mathscr{D}(I)$. *Then, there exist continuous functions* $h_i(x)$ *defined over* I *such that*

$$\langle f, \varphi\rangle = \left\langle \sum_{i=0}^{r+1} (-1)^i \left(\frac{d}{dx}\right)^i \left[\xi_{c,d}(x)\, x^{i-1} \frac{\partial h_i}{\partial x}\right],\ \varphi(x)\right\rangle$$

where $c \geqslant 0$, $d \leqslant 1$ *and* r *is an appropriate non-negative integer.*

The proof is similar tho that of Theorem 1.

Theorem 7. *For fixed* $c \geqslant 0$ *and* $d \leqslant 1$, *let* $f(t) \in \mathscr{S}'_{c,d}$ *and define*

$$F(x) \triangleq \left\langle f(t),\ \frac{t}{x^2 + t^2}\right\rangle.$$

Then, for $\varphi(x) \in \mathscr{D}(I)$,

$$\langle L_{n,x} F(x), \varphi(x) \rangle \to \langle f, \varphi \rangle \qquad as \quad n \to \infty,$$

where the operator $L_{n,x}$ *is defined by*

$$L_{n,x} = -\theta \prod_{k=1}^{n} \left(1 - \frac{\theta^{2k}}{4k^2}\right), \qquad \theta \equiv x \frac{\partial}{\partial x}.$$

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Abstract.

An infinitely differentiable and complex valued function $\varphi(x)$ *defined over* $I = (0, \infty)$ *belongs to Pandey's space* $S_\alpha(I)$ *if*

$$\gamma_k|(\varphi) = \sup_{0<x<\infty} (1+x)^\alpha \left| \left(x \frac{\partial}{\partial x}\right)^k \varphi(x) \right| < \infty,$$

for any fixed k *where* k *assumes values* 0, 1, 2, ... *and* α *is a fixed real number less than or equal to* 1. *The topology on* $S_\alpha(I)$ *is generated by the sequence of seminorms* $\{\gamma_k\}_{k=0}^{\infty}$. *Pandey extended real and complex inversion formulae of Stieltjes transforms due to Widder to* $S_\alpha(I)$*-space, but did not give a structure formula.*

In this paper an extension of $S_\alpha(I)$*-space and its dual to n-dimensions is given and a structure formula obtained which shows that every element of the dual space of* $S_\alpha(I)$ *is the linear combination of the finite order distributional derivative of certain continuous functions. The inversion formulae of Widder are also extended to another space of generalized functions.*

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