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G. P. DIKSHIT (*)

On the means of an integral function represented by Dirichlet series. (**)

1. - Let

$$f(s)=\sum_{n=1}^{\infty} a_n \exp[s\lambda_n]\,, \tag{1.1}$$

where $\lambda_1 \geqslant 0$, $\lambda_n \to \infty$, $s=\sigma+it$ be a DIRICHLET series and

$$\limsup_{n\to\infty} \frac{\log n}{\log \lambda_n} = E < \infty\,. \tag{1.2}$$

Let abscissa of convergence σ_c of (1.1) be equal to $+\infty$. Then it will represent an integral function.

The mean values $I(\sigma; f)$ and $m_k(\sigma; f)$ of the integral function $f(s)$ are defined ([**1**]$_1$, p. 52) as

$$I(\sigma; f)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f(\sigma+it)|\,\mathrm{d}t \tag{1.3}$$

and

$$m_k(\sigma; f)=\frac{2}{\exp[k\sigma]}\int_0^{\sigma} I(x; f)\exp[kx]\,\mathrm{d}x\,, \tag{1.4}$$

where $0<k<\infty$.

(*) Indirizzo: Department of Mathematics and Astronomy, Lucknow University, Lucknow, India.

(**) Ricevuto: 25-V-72.

In this paper we have studied some growth properties of the mean values defined in (1.3) and (1.4). Further, in the theorems we have taken $f(s)$ to be an integral function as defined above.

2. – Theorem 1. *Let $f(s)$ be an integral function of linear order ϱ and lower order λ. Then*

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\{m_k'(\sigma; f)/m_k(\sigma; f)\}}{\sigma} = \begin{matrix}\varrho\\ \lambda\end{matrix}, \tag{2.1}$$

where $m'(\sigma; f)$ is the derivative of $m_k(\sigma; f)$ for almost all σ.

Proof. Since $\log m_k(\sigma; f)$ is differentiable almost everywhere, we have

$$\log m_k(\sigma; f) = \log m_k(\sigma_0; f) + \int_{\sigma_0}^{\sigma} \frac{m_k'(x; f)}{m_k(x; f)}\,dx \tag{2.2}$$

for $\sigma > \sigma_0$. We know ([1]$_1$, p. 55) that $\log m_k(\sigma; f)$ is convex function of σ, therefore, $\{m_k'(\sigma; f)/m_k(\sigma; f)\}$ is increasing function for $\sigma > \sigma_0$. Thus we have

$$\log m_k(\sigma; f) \leqslant \log m_k(\sigma_0; f) + (\sigma - \sigma_0)\,\frac{m_k'(\sigma; f)}{m_k(\sigma; f)}.$$

Therefore,

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log m_k(\sigma; f)}{\sigma} \leqslant \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\{m_k'(\sigma; f)/m_k(\sigma; f)\}}{\sigma}. \tag{2.3}$$

Next, for some fixed $\eta > 0$ and $\sigma > \sigma_0$,

$$\log m_k(\sigma + \eta; f) = \log m_k(\sigma; f) + \int_{\sigma}^{\sigma+\eta} \frac{m_k'(x; f)}{m_k(x; f)}\,dx > \eta\,\frac{m_k'(\sigma; f)}{m_k(\sigma; f)}.$$

Hence

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log m_k(\sigma; f)}{\sigma} \geqslant \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\{m_k'(\sigma; f)/m_k(\sigma; f)\}}{\sigma}. \tag{2.4}$$

Combining (2.3) and (2.4) we get

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log m_k(\sigma; f)}{\sigma} = \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\{m_k'(\sigma; f)/m_k(\sigma; f)\}}{\sigma} . \tag{2.5}$$

Further, we know ([**1**]$_1$, p. 53) that

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log m_k(\sigma; f)}{\sigma} = \frac{\varrho}{\lambda} \tag{2.6}$$

and, therefore, (2.1) follows.

Theorem 2. *Let $f(s)$ be an integral function of linear order ϱ $(0<\varrho<\infty)$ and lower order λ, then*

$$\limsup_{\sigma\to\infty} \frac{\log m_k(\sigma; f)}{\sigma\, \lambda_{\nu(\sigma;f)}} \leqslant 2\left(1-\frac{\lambda}{\varrho}\right) \tag{2.7}$$

and

$$\limsup_{\sigma\to\infty} \frac{\log m_k(\sigma;f)}{\lambda_{\nu(\sigma;f)}\log \lambda_{\nu(\sigma;f)}} \leqslant 2\left(\frac{1}{\lambda}-\frac{1}{\varrho}\right) , \tag{2.8}$$

where $\nu(\sigma; f)$ is the index of the maximum term of the Dirichlet series $f(s)$ for $\operatorname{Re} s = \sigma$.

Proof. We have

$$m_k(\sigma; f) = \frac{2}{\exp[k\sigma]} \int_0^\sigma I(x; f) \exp[kx]\, dx \tag{2.9}$$

$$\leqslant \frac{2}{k} I(\sigma; f)(1-\exp[-k\sigma]) < \frac{2}{k} I(\sigma; f) .$$

Hence

$$\limsup_{\sigma\to\infty} \frac{\log m_k(\sigma; f)}{\sigma\, \lambda_{\nu(\sigma;f)}} \leqslant \limsup_{\sigma\to\infty} \frac{\log I(\sigma; f)}{\sigma\, \lambda_{\nu(\sigma;f)}} . \tag{2.10}$$

Further ($[1]_2$, p. 135)

$$\mu(\sigma; f) \leqslant I(\sigma; f) \leqslant M(\sigma; f)\,, \tag{2.11}$$

where $\mu(\sigma; f)$ and $M(\sigma; f)$ are maximum term and maximum modulus of $f(s)$ for $\operatorname{Re} s = \sigma$. Also under the condition (1.2) and for integral functions of finite linear order, we know that

$$\log \mu(\sigma; f) \sim \log M(\sigma; f)\,, \tag{2.12}$$

therefore

$$\limsup_{\sigma\to\infty} \frac{\log \mu(\sigma; f)}{\sigma \lambda_{\nu(\sigma; f)}} = \limsup_{\sigma\to\infty} \frac{\log I(\sigma; f)}{\sigma \lambda_{\nu(\sigma; f)}} = \limsup_{\sigma\to\infty} \frac{\log M(\sigma; f)}{\sigma \lambda_{\nu(\sigma; f)}}\,. \tag{2.13}$$

It is known ([2], p. 84) that

$$\limsup_{\sigma\to\infty} \frac{\log \mu(\sigma; f)}{\sigma \lambda_{\nu(\sigma; f)}} \leqslant 2\left(1 - \frac{\lambda}{\varrho}\right) \tag{2.14}$$

and

$$\limsup_{\sigma\to\infty} \frac{\log \mu(\sigma; f)}{\lambda_{\nu(\sigma; f)} \log \lambda_{\nu(\sigma; f)}} \leqslant 2\left(\frac{1}{\lambda} - \frac{1}{\varrho}\right). \tag{2.15}$$

Combining (2.10), (2.13) and (2.14) we get (2.7). Similarly by using (2.15) and proceeding on the same lines (2.8) follows.

3. – Theorem 3. *Let $f(s)$ be an integral function of linear order ϱ $(0 < \varrho < \infty)$, type τ and lower type t, then*

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log m_k(\sigma; f)}{\exp[\varrho\sigma]} = \frac{\tau}{t}\,. \tag{3.1}$$

Proof. Using (2.9), we get

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log m_k(\sigma; f)}{\exp[\varrho\sigma]} \leqslant \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log I(\sigma; f)}{\exp[\varrho\sigma]}\,.$$

Further, (2.11) and (2.12) lead to

$$\log I(\sigma; f) \sim \log M(\sigma; f)\,.$$

Hence

$$(3.2)\qquad \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log m_k(\sigma; f)}{\exp[\varrho\sigma]} \leqslant \frac{\tau}{t}.$$

Again, for $\eta > 0$ and $\sigma > \sigma_0$

$$m_k(\sigma+\eta; f) = \frac{2}{\exp[k(\sigma+\eta)]}\int_0^{\sigma+\eta} \exp[kx] I(x; f)\,dx$$

$$\geqslant \frac{2}{\exp[k(\sigma+\eta)]}\int_\sigma^{\sigma+\eta} I(x; f)\exp[kx]\,dx$$

$$\geqslant \frac{2}{k} I(\sigma; f)(1-\exp[-k\eta]),$$

for $I(\sigma; f)$ is increasing function of σ for $\sigma > \sigma_0$. Further, using (2.11) leads to

$$m_k(\sigma+\eta; f) \geqslant \frac{2}{k}\mu(\sigma; f)(1-\exp[-k\eta]) .$$

Hence

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log m_k(\sigma; f)}{\exp[\varrho\sigma]} \geqslant \exp[-\varrho\eta] \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \mu(\sigma; f)}{\exp[\varrho\sigma]} .$$

Since left hand side is independent of η, on taking $\eta\to 0$ we get

$$(3.3)\qquad \lim_{\varrho\leftarrow\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log m_k(\sigma.; f)}{\exp[\varrho\sigma]} \geqslant \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \mu(\sigma; f)}{\exp[\varrho\sigma]} = \frac{\tau}{t}.$$

Combining (3.2) and (3.3) leads to (3.1).

Theorem 4. *Let $f(s)$ be an integral function. Then*

$$(3.4)\qquad \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{m_k(\sigma; f)}{M(\sigma; f)} \leqslant \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{m_k(\sigma; f)}{I(\sigma; f)} \leqslant \frac{2}{k}.$$

Proof.

$$m_k(\sigma; f) = \frac{2}{\exp[k\sigma]}\int_0^{\sigma} I(x; f)\exp[kx]\,dx \leqslant I[\sigma; f)\frac{2}{k}(1-\exp[-k\sigma]) .$$

Therefore, we have

$$\limsup_{\sigma\to\infty} \frac{m_k(\sigma; f)}{I(\sigma; f)} \leqslant \frac{2}{k}. \tag{3.5}$$

Again from (2.11), we have

$$I(\sigma; f) \leqslant M(\sigma; f).$$

Hence

$$\limsup_{\sigma\to\infty} \frac{m_k(\sigma; f)}{M(\sigma; f)} \leqslant \limsup_{\sigma\to\infty} \frac{m_k(\sigma; f)}{I(\sigma; f)} \leqslant \frac{2}{k},$$

which is (3.4).

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References.

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