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Holomorphic solutions of singular Darboux problems. (**)

Introduction.

A. F. MARTINOLLI [4] considered a singular DARBOUX problem

$$(1.1)\quad \begin{cases} xy\dfrac{\partial^2 u}{\partial x\,\partial y} = xA(x, y)\dfrac{\partial u}{\partial x} + yB(x, y)\dfrac{\partial u}{\partial y} + C(x, y)u + xy\, f\left(x, y, u, \dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}\right) \\ u(x, 0) = \varphi(x)\,, \qquad u(0, y) = \Psi(y)\,, \end{cases}$$

in a rectangle $\{(x, y) \in \mathfrak{R}^2;\ 0 \leqq x \leqq a,\ 0 \leqq y \leqq b\}$ in the plane $\mathfrak{R}^2$ of two real variables x and y. In this paper we shall treat the case that $A(x, y)$, $B(x, y)$, $C(x, y)$, $f(x, y, u, p, q)$, $\varphi(x)$ and $\Psi(y)$ are holomorphic functions of complex variables. We shall seek holomorphic solutions of the problem (1.1) in a neighborhood of the origin in the space $\mathfrak{C}^2$ of two complex variables x and y and discuss the existence and uniqueness of a real-valued real analytic solution $u(x, y)$ in a neighborhood of the origin of the singular elliptic equation

$$(1.2)\quad (x^2 + y^2)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 4A(x, y)\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) + 4B(x, y)u + $$
$$+ 4(x^2 + y^2)\left(C(x, y) + D(x, y)\frac{\partial u}{\partial x} + E(x, y)\frac{\partial u}{\partial y}\right)$$

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as an application, where A, B, C, D and E are real-valued real analytic functions in a neighborhood of the origin in $\mathfrak{R}^2$.

1. - Integral equations.

Let $A(x, y)$, $B(x, y)$ and $C(x, y)$ be holomorphic functions in a didisc $U_R = \{(x, y) \in \mathfrak{C}^2;\ |x| < R,\ |y| < R\}$ satisfying

$$|A(x, y)| \leqq M\,, \qquad |B(x, y)| \leqq M\,, \qquad |C(x, y)| \leqq M \tag{1.3}$$

in U_R. Let $\varphi(x)$ and $\Psi(y)$ be, respectively, holomorphic functions in $|x| < R$ and $|y| < R$ satisfying

$$\varphi(0) = 0\,,\ |\varphi(x)| \leqq m\,,\ |\varphi'(x)| \leqq m\,,\ \Psi(0) = 0\,,\ |\Psi(y)| \leqq m\,,\ |\Psi'(y)| \leqq m\,. \tag{1.4}$$

Let $f(x, y, u, p, q)$ be a holomorphic function in

$$F = \{(x, y, u, p, q) \in \mathfrak{C}^5;\ |x| < R,\ |y| < R,\ |u - \varphi(x) - \Psi(y)| < r,\ |p - \varphi'(x)| < r,\ |q - \Psi'(y)| < r\}$$

satisfying

$$\begin{cases} |f(x, y, u, p, q)| \leqq M\,, \\ |f(x, y, u_1\ p_1\ q_1) - f(x, y, u_2\ p_2\ q_2)| \leqq \\ \qquad \leqq M\big(|u_1 - u_2| + |p_1 - p_2| + |q_1 - q_2|\big)\,, \end{cases} \tag{1.5}$$

for (x, y, u, p, q), (x, y, u_1, p_1, q_1), $(x, y, u_2, p_2, q_2) \in F$.

Lemma 1. *If $u(x, y)$ is a holomorphic solution of (1.1) in U_R satisfying $(x, y, u, \partial u/\partial x, \partial u/\partial y) \in F$ for $(x, y) \in U_R$, then we have*

$$xA(x, 0)\varphi'(x) + C(x, 0)\varphi(x) = 0\,, \qquad \varphi(0) = 0 \tag{1.6}$$

and

$$yB(0, y)\Psi'(y) + C(0, y)\Psi(y) = 0\,, \qquad \Psi(0) = 0\,. \tag{1.7}$$

Moreover we have

$$u(x, y) = \varphi(x) + \Psi(y) + \tag{1.8}$$

$$+ \int_0^x ds \int_0^y \left\{ \frac{sA(s, t) \dfrac{\partial u(s, t)}{\partial s} + tB(s, t) \dfrac{\partial u(s, t)}{\partial t} + C(s, t)\, u(s, t)}{st} + \right.$$

$$\left. + f\left(s, t, u(s, t), \frac{\partial u(s, t)}{\partial s}, \frac{\partial u(s, t)}{\partial t}\right)\right\} dt .$$

Proof. Substituting $y = 0$ and $x = 0$ in (1.1) respectively, we have (1.6) and (1.7). Since $(\partial u/\partial y)(0, y) = \Psi'(y)$ and $\Psi(0) = 0$, we have (1.8) by direct integration of $\partial^2 u/\partial x\, \partial y$.

Lemma 2 (SCHWARZ Lemma). *Let $a(x, y)$ be a holomorphic function in U_R satisfying $a(x, 0) = 0$, $a(0, y) = 0$ and*

$$|a(x, y)| \leqq L$$

in U_R. Then we have

$$|a(x, y)| \leqq \frac{L}{R^2} \tag{1.9}$$

in U_R.

Proof. There is a holomorphic function $b(x, y)$ in U_R such that $a(x, y) = xyb(x, y)$ in U_R. Let ϱ be a positive number with $\varrho < R$. By the maximum modulus principle, we have

$$\operatorname*{Max}_{|x| \leqq \varrho,\ |y| \leqq \varrho} |b(x, y)| \leqq \operatorname*{Max}_{|x| = \varrho,\ |y| = \varrho} |b(x, y)| = \frac{\operatorname*{Max}_{|x| = \varrho,\ |y| = \varrho} |a(x, y)|}{\varrho^2} \leqq \frac{L}{\varrho^2} .$$

Hence we have $|b(x, y)| \leqq L/R^2$ in U_R.

Lemma 3. *Assume that $6M < 1$. Then the existence of a solution of the problem* (1.8), *which is holomorphic in a neighborhood of the origin, is unique.*

Proof. Let $u(x, y)$ and $v(x, y)$ be holomorphic solutions of (1.8) in U_δ.

We shall prove that there holds $u(x, y) = v(x, y)$ identically. By the theorem of identity, we may assume that $0 < \delta < 1$. We put

$$w(x, y) = u(x, y) - v(x, y) .$$

Then $w(x, y)$ satisfies

$$(1.10) \quad w(x, y) = \int_0^x ds \int_0^y \Bigg\{ \frac{sA(s, t)\dfrac{\partial w(s, t)}{\partial s} + tB(s, t)\dfrac{\partial w(s, t)}{\partial t} + C(s, t)w(s, t)}{st} + \\ + f\left(s, t, u(s, t), \frac{\partial u(s, t)}{\partial s}, \frac{\partial u(s, t)}{\partial t}\right) - \\ - f\left(s, t, v(s, t), v(s, t), \frac{\partial y(s, t)}{\partial s}, \frac{\partial y(s, t)}{\partial t}\right)\Bigg\} dt .$$

We put

$$\varepsilon = \operatorname*{Max}_{\substack{|x| < \delta \\ |y| < \delta}} \operatorname{Max} \left(|w(x, y)|, \left| x \frac{\partial w(x, y)}{\partial x} \right|, \left| y \frac{\partial w(x, y)}{\partial y} \right| \right) .$$

By the maximum modulus principle we have

$$\left| \frac{\partial w(x, y)}{\partial x} \right| \leqq \frac{\varepsilon}{\delta}, \qquad \left| \frac{\partial w(x, y)}{\partial y} \right| \leqq \frac{\varepsilon}{\delta}$$

in U_δ. Since

$$w(x, 0) = (\partial w/\partial x)(x, 0) = 0 , \qquad w(0, y) = (\partial w/\partial y)(0, y) = 0 ,$$

by Lemma 2 we have

$$\left| \frac{sA(s, t)\dfrac{\partial w(s, t)}{\partial s} + tB(s, t)\dfrac{\partial w(s, t)}{\partial t} + C(s,t)w(s, t)}{st} \right| \leqq \frac{3M\varepsilon}{\delta^2}$$

in U_δ. By (1.5) we have

$$\left| f\left(s,t,u(s,t),\frac{\partial u(s,t)}{\partial s},\frac{\partial u(s,t)}{\partial t}\right) - f\left(s,t,v(s,t),\frac{v(s,t)}{s},\frac{\partial y(s,t)}{\partial t}\right)\right| \leqq \frac{3M\varepsilon}{\delta}.$$

By (1.10) we have $|w(x,y)|\leqq 6M\varepsilon$ in U_δ. Similarly, we have

$$\left| x\,\frac{\partial w(x,y)}{\partial x}\right| \leqq 6M\varepsilon\,, \qquad \left| y\,\frac{\partial w(x,y)}{\partial y}\right| \leqq 6M\varepsilon\,.$$

Hence we have $\varepsilon\leqq 6M\varepsilon$. Since $6M<1$, we have $\varepsilon=0$.

In case that $M=1$, the existence of a solution of the problem (1.8) is not necessarily unique as we will give an example in the last paragraph of this paper.

2. - Linear equations.

Let $D(x,y)$ be a holomorphic function in U_R satisfying $|D(x,y)|\leqq L$ in U_R. Consider the linear problem

$$\text{(2.1)}\qquad \begin{cases} xy\,\dfrac{\partial^2 u}{\partial x\,\partial y} = xA(x,y)\,\dfrac{\partial u}{\partial x} + yB(x,y)\,\dfrac{\partial u}{\partial y} + C(x,y)u + xyD(x,y)\,, \\[2ex] u(x,0)=\varphi(x)\,, \qquad u(0,y)=\Psi(y)\,. \end{cases}$$

Let $\mathfrak{H}_R$ be the set of all holomorphic functions in U_R which satisfies

$$\|v\|_R = \underset{(x,y)\in U_R}{\text{Max}}\ \text{Max}\left(\left|\frac{v(x,y)}{xy}\right|,\left|\frac{1}{y}\,\frac{\partial v(x,y)}{\partial x}\right|,\left|\frac{1}{x}\,\frac{\partial v(x,y)}{\partial y}\right|\right) < +\infty\,.$$

Lemma 4. *For* $v\in\mathfrak{H}_R$ *we put*

$$\text{(2.2)}\qquad (Sv)(x,y)=\int_0^x \mathrm{d}s \int_0^y \left\{\frac{sA(s,t)\,\dfrac{\partial v(s,t)}{\partial s} + tB(s,t)\,\dfrac{\partial v(s,t)}{\partial s} + C(s,t)v(s,t)}{st}\right\}\mathrm{d}t$$

in U_R. *Then* $Sv\in\mathfrak{H}_R$ *and we have*

$$\text{(2.3)}\qquad \|Sv\|_R \leqq 3M\|v\|_R\,.$$

Proof. Since we have

$$\left| \frac{sA(s,t)\dfrac{\partial v(s,t)}{\partial s} + tB(s,t)\dfrac{\partial v(s,t)}{\partial t} + C(s,t)v(s,t)}{st} \right| \leqq 3M\,\|v\|_R\,,$$

we have $|(Sv)(x,y)| \leqq 3M\|v\|_R|xy|$ in U_R by (2.2). Similarly we have

$$\left|\frac{\partial(Sv)}{\partial x}(x,y)\right| \leqq 3M\|v\|_R|y|\,, \quad \left|\frac{\partial(Sv)}{\partial y}(x,y)\right| \leqq 3M\|v\|_R|x|$$

in U_R. Hence we have (2.3).

For $v \in \mathfrak{H}_R$ we put

$$(2.4) \qquad (Tv)(x,y) = (Sv)(x,y) + \int_0^x \mathrm{d}s \int_0^y D(s,t)\,\mathrm{d}t\,.$$

Lemma 5. *For* $v \in \mathfrak{H}_R$, $Tv \in \mathfrak{H}_R$. $T(\varphi(x) + \Psi(y))$ *belongs to* $\mathfrak{H}_R$ *and satisfies*

$$(2.5) \qquad \|T(\varphi + \Psi)\|_R \leqq \left(\frac{2(R+1)Mm}{R^2} + L\right)\,.$$

Proof. Since

$$\chi(s,t) = sA(s,t)\varphi'(s) + tB(s,t)\Psi'(t) + C(s,t)(\varphi(s) + \Psi(t)) = 0$$

in U_R when $s = 0$ or $t = 0$, by Lemma 2 and (1.4) we have

$$\left|\frac{\chi(s,t)}{st}\right| \leqq \frac{2(R+1)Mm}{R^2}$$

in U_R. Hence we have (2.5).

Proposition 6. *Assume that* $6M < 1$. *Then the problem* (2.1) *has a unique holomorphic solution* $u(x,y)$ *in* U_R; $u(x,y) - \varphi(x) - \Psi(y)$ *belongs to* $\mathfrak{H}_R$ *and satisfies*

$$(2.6) \qquad \|u - \varphi - \Psi\|_R \leqq 2\left(\frac{2(R+1)Mm}{R^2} + L\right)\,.$$

Proof. The problem (2.1) is equivalent to the integral equation

$$u(x, y) = \varphi(x) + \Psi(y) + (Tu)(x, y)\,. \tag{2.7}$$

We shall solve the integral equation (2.7) by the method of successive approximations:

$$u_{n+1}(x, y) = \varphi(x) + \Psi(y) + (Tu_n)(x, y) \qquad (n \geqq 1)\,,$$
$$u_1(x, y) \quad = \varphi(x) + \Psi(y)\,,$$

in U_R. By Lemmas 4 and 5 $\{u_n(x, y)\}$ can be defined successively so as to be a sequence of holomorphic functions in U_R. We put

$$v_n(x, y) = u_{n+1}(x, y) - u_n(x, y) \qquad (n \geqq 1)\,.$$

Then we have

$$v_n(x, y) = (Sv_{n-1})(x, y) \qquad (n \geqq 2)\,.$$

There holds $v_1(x, y) = \big(T(\varphi + \Psi)\big)(x, y)$.

By Lemmas 4 and 5, we have

$$\|v_n\|_R \leqq (3M)^{n-1} \left(\frac{2(R+1)\,Mm}{R^2} + L \right).$$

Since $6M < 1$, $\{u_n(x, y)\}$ converges uniformly to a holomorphic solution $u(x, y)$ of (2.1) in U_R which is a unique solution of (2.1) by Lemma 3. Moreover we have (2.6).

3. - Non-linear equations.

Now we will return to the problem (1.1), that is, the integral equation (1.8). Assume that $6M < 1$. We shall solve it by the method of successive approximations:

$$\left\{\begin{array}{l} xy\, \dfrac{\partial^2 u_{n+1}}{\partial x\, \partial y} = xA\, \dfrac{\partial u_{n+1}}{\partial x} + yB\, \dfrac{\partial u_{n+1}}{\partial y} + Cu_{n+1} + xy\, f\left(x, y, u_n, \dfrac{\partial u_n}{\partial x}, \dfrac{\partial u_n}{\partial y}\right) \\ u_{n+1}(x, 0) = \varphi(x)\,, \qquad u_{n+1}(0, y) = \Psi(y)\,, \qquad (n \geqq 1)\,, \end{array}\right. \tag{3.1}$$

$$u_1(x, y) = \varphi(x) + \Psi(y)\,. \tag{3.2}$$

Assume that $u_1(x, y), u_2(x, y), \dots u_n(x, y)$ are well-defined in U_δ for suitable $\delta > 0$ with $\delta \leqq 1$ and $\delta \leqq R$. We put

$$v_p(x, y) = u_{p+1}(x, y) - u_p(x, y) \qquad (p = 1, 2, \dots\ n-1)\ .$$

Then $v_p(x, y)$ is a unique holomorphic solution of the problem

$$(3.3)\quad \begin{cases} xy\ \dfrac{\partial^2 v_p(x, y)}{\partial x\, \partial y} = xA\ \dfrac{\partial v_p(x, y)}{\partial x} + yB\ \dfrac{\partial v_p(x, y)}{\partial y} + Cv_p(x, y) + \\ \quad + xy\left\{f\left(x, y, u_p, \dfrac{\partial u_p}{\partial x}, \dfrac{\partial u_p}{\partial y}\right) - f\left(x, y, u_{p-1}, \dfrac{\partial u_{p-1}}{\partial x}, \dfrac{\partial u_{p-1}}{\partial y}\right)\right\} \\ v_p(x, 0) = 0\ , \qquad v_p(0, y) = 0\ , \end{cases}$$

for $p = 2, 3, \dots\ n-1$. The problem (3.1) has also a holomorphic solution in U_R for $n = 2$ such that $v_1 = u_2 - \varphi - \Psi \in \mathfrak{H}_R$ and

$$\|v_1\|_R \leqq \frac{1}{3}\left(\frac{2(R+1)m}{R^2} + 1\right)$$

since $6M < 1$. Assume that $v_{p-1} \in \mathfrak{H}_R$ and

$$\|v_{p-1}\| < \frac{\delta^{p-2}}{3}\left(\frac{2(R+1)m}{R^2} + 1\right)$$

for $p = 2, 3, \dots\ n$. Then we have

$$\left| f\left(x, y, u_p, \frac{\partial u_p}{\partial x}, \frac{\partial u_p}{\partial y}\right) - f\left(x, y, u_{p-1}, \frac{\partial u_{p-1}}{\partial x}, \frac{\partial u_{p-1}}{\partial y}\right)\right| \leqslant$$

$$\leqq M\left(|v_{p-1}| + \left|\frac{\partial v_{p-1}}{\partial x}\right| + \left|\frac{\partial v_{p-1}}{\partial y}\right|\right) \leqq 3M\ \frac{\delta^{p-2}}{3}\left(\frac{2(R+1)m}{R^2} + 1\right)\delta$$

$$< \frac{\delta^{p-1}}{6}\left(\frac{2(R+1)m}{R^2} + 1\right),$$

since $6M < 1$. By Proposition 6, we have $v_p \in \mathfrak{H}_\delta$ and

$$(3.4)\qquad \|v_p\|_\delta < \frac{\delta^{p-1}}{3}\left(\frac{2(R+1)m}{R^2} + 1\right).$$

Thus we have proved (3.4) for $p=1,2,\dots.\ n$.

Now we choose δ so as to satisfy

$$0<\delta\leqq \operatorname{Min}\left\{\frac{\varepsilon r}{R,\ [2(R+1)m/R^2]-1+3r}\right\}. \tag{3.5}$$

Then we have

$$|u_p(x,y)-u_1(x,y)|,\ \left|\frac{\partial u_p}{\partial x}-\varphi'(x)\right|,\ \left|\frac{\partial u_p}{\partial y}-\Psi'(y)\right|<\frac{\dfrac{2(R+1)m}{R^2}+1}{3(1-\delta)}\cdot\delta\leqq r$$

for $p=1,2,\dots,n+1$. Thus for a positive number δ satisfying (3.5) $\{u_n(x,y)\}$ can be defined in U_δ and converge uniformly to a unique holomorphic solution $u(x,y)$ of the problem (1.1). We summarize the above result in the following theorem.

Theorem 1. *Let $A(x,y)$, $B(x,y)$ and $C(x,y)$ be holomorphic functions in $\{(x,y)\in\mathfrak{C}^2;\ |x|<R,\ |y|<R\}$ satisfying*

$$|A(x,y)|\leqq M,\qquad |B(x,y)|\leqq M,\qquad |C(x,y)|\leqq M.$$

Let $\varphi(x)$ and $\Psi(y)$ be, respectively, holomorphic functions in $\{x\in\mathfrak{C};\ |x|<R\}$ and $\{y\in\mathfrak{C},\ |y|<R\}$ satisfying

$$xA(x,0)\varphi'(x)+C(x,0)\varphi(x)=0,\qquad \varphi(0)=0 \tag{3.4}$$

nd

$$yB(0,y)\Psi'(y)+C(0,y)\Psi(y)=0,\qquad \Psi(0)=0. \tag{3.5}$$

Moreover, let $f(x,y,u,p,q)$ be a holomorphic function in

$$\{(x,y,u,p,q)\in\mathfrak{C}^5;\quad |x|<R,\ |y|<R,\ |u-\varphi(x)-\Psi(y)|<r,$$
$$|p-\varphi'(x)|<r,\ |q-\Psi'(y)|<r\}$$

satisfying

$$\begin{cases} |f(x,y,u,p,q)|\leqq M,\\ |f(x,y,u_1,p_1,q_1)-f(x,y,u_2,p_2,q_2)|\leqq\\ \qquad\leqq M(|u_1-u_2|+|p_1-p_2|+|q_1-q_2|). \end{cases} \tag{3.6}$$

Assume that $6M < 1$. *Then for sufficiently small* $\delta > 0$, *the problem* (1.1) *has a unique holomorphic solution in* $\{(x, y) \in \mathfrak{C}^2;\ |x| < \delta,\ |y| < \delta\}$.

Theorem 2. *Let* $A(x, y)$, $B(x, y)$ *and* $C(x, y)$ *be holomorphic functions given in Theorem* 1. *Let* $f(x, y, u, p, q)$ *be a holomorphic function in*

$$\{(x, y, u, p, q) \in \mathfrak{C}^5;\ |x| < R,\ |y| < R,\ |u| < r,\ |p| < r,\ |q| < r\}$$

satisfying the condition (3.6). *Assume that* $6M < 1$, $(AB)(0, 0) \neq 0$, $(C/A)(0, 0) \neq$ *non-positive integer and* $(C/B)(0, 0) \neq$ *non-positive integer. Then for* $\delta = \mathrm{Min}\,(R, 3r/(1 + 3r))$ *the singular hyperbolic equation*

$$xy \frac{\partial^2 u}{\partial x \partial y} = xA \frac{\partial u}{\partial x} + yB \frac{\partial u}{\partial y} + Cu + xyf\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \tag{3.7}$$

has a unique holomorphic solution $u(x, y)$ *in* $\{(x, y) \in \mathfrak{C}^2;\ |x| < \delta,\ |y| < \delta\}$.

Proof. Let $u(x, y)$ be a holomorphic solution of (3.7). Then $\varphi(x) = u(x, 0)$ and $\Psi(y) = u(0, y)$ satisfy (3.4) and (3.5) respectively. By the assumption of Theorem 2, φ and Ψ are identically zero. We have Theorem 2 by Theorem 1.

4. - Singular elliptic equations.

Let $A(x, y)$, $B(x, y)$, $C(x, y)$, $D(x, y)$ and $E(x, y)$ be real-valued real analytic functions of real variables x and y in a neighborhood of the origin in $\mathfrak{R}^2$. Consider a singular elliptic equation

$$\left\{\begin{aligned} &(x^2 + y^2)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 4A\,(x, y)\left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y}\right) + 4B(x, y)u + \\ &+ 4(x^2 + y^2)\left(C(x, y) + D(x, y)\ \frac{\partial u}{\partial x} + E(x, y)\ \frac{\partial u}{\partial y}\right). \end{aligned}\right. \tag{4.1}$$

For sufficiently small $r > 0$, A, B, C, D and E can be extended to holomorphic functions in $U_r = \{(x, y) \in \mathfrak{C}^2;\ |x| < r,\ |y| < r\}$ of two complex variables x and y satisfying

$$|A(x, y)| \leqq M\,, \qquad |B(x, y)| \leqq M\,,$$

$$|C(x, y)| \leqq L\,, \qquad |D(x, y)| \leqq L\,, \qquad |E(x, y)| \leqq L\,.$$

Let $u(x, y)$ be a real-valued real analytic solution of the equation (4.1) in a neighborhood of the origin in $\mathfrak{R}^2$; $u(x, y)$ can be extended to a holomorphic function in U_r sufficiently small r.

Conversely let $u(x, y)$ be a holomorphic solution of the equation (4.1). The real part of u is a real-valued real analytic solution of the equation (4.1) when we regard independent variables x and y as real variables. So we had better consider the equation (4.1) in a domain in the space $\mathfrak{C}^2$ of two complex variables x and y, even if our aim is to seek real-valued real analytic solutions of the equation (4.1) with respect to real variables x and y.

So we regard x and y as complex variables and perform changement of independent complex variables:

$$\begin{cases} z = x + iy \\ \zeta = x - iy \end{cases} \qquad \begin{cases} x = \dfrac{z+\zeta}{2} \\ y = \dfrac{z-\zeta}{2i} . \end{cases}$$

Then the equation (4.1) become

$$(4.2) \quad z\zeta \frac{\partial^2 u}{\partial z \partial \zeta} = A\left(z\frac{\partial u}{\partial z} + \zeta \frac{\partial u}{\partial \zeta}\right) + Bu + z\zeta \left(C + (D+iE)\frac{\partial u}{\partial z} + (D-iE)\frac{\partial u}{\partial \zeta}\right) .$$

Let $u(x, y)$ be a holomorphic solution of the equation (4.1) with $u(0, 0) = 0$. We put

$$(4.3) \qquad \varphi(z) = u\left(\frac{z}{2}, \frac{z}{2i}\right), \quad \Psi(\zeta) = u\left(\frac{\zeta}{2}, \frac{\zeta}{2i}\right) .$$

Then we have

$$(4.4) \qquad zA\left(\frac{z}{2}, \frac{z}{2i}\right)\varphi'(z) + B\left(\frac{z}{2}, \frac{z}{2i}\right)\varphi(z) = 0 , \qquad \varphi(0) = 0 ,$$

$$(4.5) \qquad \zeta A\left(\frac{\zeta}{2}, -\frac{\zeta}{2i}\right)\Psi'(\zeta) + B\left(\frac{\zeta}{2}, -\frac{\zeta}{2i}\right)\Psi(\zeta) = 0 , \quad \Psi(0) = 0 .$$

By Proposition 6, we have the following proposition.

Proposition 7. *Assume that* $6M < 1$. *For any holomorphic function* $\varphi(z)$ *and* $\Psi(\zeta)$ *in* $\{z \in \mathfrak{C};\ |z| < r\}$ *and* $\{\zeta \in \mathfrak{C};\ |\zeta| < r\}$ *satisfying* (4.4) *and* (4.5) *respectively, the problem*

$$(4.6)\quad \begin{cases} (x^2+y^2)\left(\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}\right) = 4A(x, y)\left(x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}\right) + 4B(x, y)u + \\[2ex] + 4B(x^2+y^2)\left(C(x, y) + D(x, y)\dfrac{\partial u}{\partial x} + E(x, y)\dfrac{\partial u}{\partial y}\right), \\[2ex] u\left(\dfrac{z}{2}, \dfrac{z}{2i}\right) = \varphi(z), \qquad u\left(\dfrac{\zeta}{2}, \dfrac{\zeta}{2i}\right) = \Psi(\zeta), \end{cases}$$

has a unique holomorphic solution $u(x, y)$ *in* $U_{r/2}$.

By Theorem 2, (4.4) and (4.5), we have the following proposition.

Proposition 8. *Assume that* $6M < 1$, $A(0, 0) \neq 0$ *and* $(B/A)(0, 0) \neq$ *non-positive integer. Then the singular elliptic equation* (4.1) *has a unique real-valued real analytic solution* $u(x, y)$ *in* $\{(x, y) \in \mathfrak{R}^2;\ |x| < r/2,\ |y| < r/2\}$.

Example. Let A and B be real numbers. Let us seek real-valued real analytic solutions of the singular elliptic equation

$$(4.7)\qquad (x^2+y^2)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) = 4Au + 4(x^2+y^2)B.$$

At first consider the case $B \neq 0$. Let $u(x, y)$ be its solution in a neighborhood of the origin and let

$$(4.8)\qquad u(x, y) = \sum_{n=0}^{\infty} u_n(x, y)$$

be the homogeneous polynomial series expansion of $u(x, y)$. For $n \neq 2$ each homogeneous polynomial $u_n(x, y)$ of degree n is a solution of the homogeneous equation

$$(4.9)\qquad (x^2+y^2)\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) = 4Au$$

and $u_2(x, y)$ is a solution of the equation (4.7). When $B \neq 0$ the equation (4.7) has a homogeneous polynomial solution of degree 2 if an only if $A \neq 1$. If $B \neq 0$ and $A \neq 1$,

$$u(x, y) = \frac{B(x^2 + y^2)}{1 - A} \tag{4.10}$$

is a homogeneous polynomial solution of degree 2.

When $B = 0$, the equation (4.7) has a homogeneous polynomial solution of degree 2 if and only if $A = 1$. When $B = 0$ and $A = 1$, the polynomial defined by

$$u(x, y) = a(x^2 + y^2) \tag{4.11}$$

is a solution of the equation (4.7) for any real number $a \neq 0$. When $A = 0$, the set of all solutions of the equation (4.9) is precisely the set of all functions $u(x, y)$ which are harmonic in neighborhoods of the origin.

Let

$$u(x, y) = \sum_{p=0}^{n} a_p x^p y^{n-p}$$

be a homogeneous polynomial solution of degree n of the equation (4.9). We introduce polar coordinates by

$$x = r \cos\theta\,, \qquad y = r \sin\theta\,. \tag{4.12}$$

Then we have $u(r\cos\theta, r\sin\theta) = r^n f(\theta)$ where

$$f(\theta) = \sum_{\nu=0}^{n} a_\nu \cos^\nu\theta \sin^{n-\nu}\theta\,. \tag{4.13}$$

$f(\theta)$ satisfies

$$f''(\theta) + (n^2 - 4A) f(\theta) = 0\,. \tag{4.14}$$

(4.14) has a non-trivial solution of type (4.13) only when $4A = n^2 - p^2$ for an integer p with $0 \leqq p \leqq n$. Then the solution of the equation (4.14) is of the form

$$f(\theta) = C \exp[ip\theta] + D \exp[-ip\theta]\,. \tag{4.15}$$

By (4.12) we have

$$(4.16)\qquad r^p \exp[ip\theta] = (x+iy)^p\,, \qquad r^p \exp[-ip\theta] = (x-iy)^p\,.$$

Substituting (4.16) in $u(r\cos\theta, r\sin\theta)$, we have

$$(4.17)\qquad u(x, y) = (x^2+y^2)^{(n-p)/2}\big(C(x+iy)^p + D(x-iy)^p\big)\,.$$

In order that $u(x, y)$ given by (4.17) is a homogeneous polynomial solution of degree n of the equation (4.9), it is necessary and sufficient that $n-p = 2q$ for a non-negative integer q. Let $X_{n,q}(x, y)$ and $Y_{n,q}(x, y)$ be, respectively, the real and imaginary parts of $(x+iy)^{n-2q}$ when x and y are regarded as real variables. Then

$$(4.18)\qquad u(x, y) = (x^2+y^2)^q\big(bX_{n,q}(x, y) + cY_{n,q}(x, y)\big)$$

is a real-valued homogeneous polynomial solution of degree n of the equation (4.9) for any non-zero real vector (c, d). Summarizing the above result, we have the following proposition.

Proposition 9. *There occur four cases.*

(1) $A=1$, $B\neq 0$. *There are no solutions of the equation* (4.7).

(2) $A=1$, $B=0$. *The function given by* (4.11) *is a real-valued real analytic solution of the equation* (4.7) *for any real number* a.

(3) $A\neq 1$, $4A = n^2-(n-2q)^2$ *for some pair* (n, q) *of non-negative integers* n *and* q *with* $2q\leqq n$. *Any real analytic function* $u(x, y)$ *of the form*

$$(4.19)\qquad u(x, y) = \frac{B(x^2+y^2)}{1-A} + \sum_{\substack{4A=n^2-(n-2q)^2\\ 2q\leqq n}} (x^2+y^2)^q\big(b_{n,q}X_{n,q}(x,y) + c_{n,q}Y_{n,q}(x,y)\big)\,,$$

is a real-valued real analytic solutions of the equation (4.7) *for any real numbers* $b_{n,q}$ *and* $c_{n,q}$ *such that* (4.19) *converges in a neighborhood of the origin.*

(4) $A\neq 1$, $4A\neq n^2-(n-2q)^2$ *for any pair* (n, q) *of non-negative integers* n *and* q *with* $2q\leqq n$. *The function defined by* (4.10) *is a unique solution of the equation* (4.7).

In other words, there is a discrete subset Δ of $\mathfrak{R}$ such that the equation (4.7) has a unique real-valued real analytic solution (4.10) for $A\notin\Delta$.

This means that some restriction on M can not be omitted in Proposition 6 and 7, although the condition $6M < 1$ is not a sharp one.

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Riassunto.

L'esistenza di soluzioni olomorfe di problemi singolari di Darboux è studiata.

Résumé.

L'existence des solutions holomorphes des problèmes singuliers de Darboux est étudiée.

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