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Integral equations of convolution form. (**)

Introduction.

Integral equations of the form

$$\int_0^t \varphi(t-x) f(x)\, dx = g(t),$$

where f or φ is the unknown function, are of immense importance due to their frequent occurrence in applied mathematics. This paper gives the solutions of three such equations using LAPLACE transform. These results generalize theorems earlier given by JET WIMP ([4], p. 44) and RUSIA ([6]$_1$, pp. 119, 123, 127, 129; [6]$_2$, pp. 16, 17; [7]$_3$, pp. 67-70).

1. - Definitions and results used.

If

$$\varphi(p) = \int_0^\infty \exp[-pt] f(t)\, dt, \tag{1.1}$$

then $\varphi(p)$ is called the LAPLACE transform of $f(t)$ and $f(t)$ is called the inverse LAPLACE transform of $\varphi(p)$ and this relation is denoted by

$$\varphi(p) \doteq f(t).$$

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Re (a) denotes the real part of the complex number a.
$\Delta(k, s)$ denotes the set of k parameters

$$s/k,\ (s+1)/k,\ (s+2)/k,\ \ldots,\ (s+k-1)/k\ .$$

${}_1(a_j)_n$ denotes the set of n parameters $a_1, a_2, \ldots, a_n$.

${}_1(a_j, e_j)_n$ denotes the set of n pairs of parameters

$$(a_1, e_1),\ (a_2, e_2),\ \ldots,\ (a_n, e_n)\ .$$

(1.2) $$ {}_p\psi_q[{}_1(a_j, e_j)_p;\ {}_1(b_j, f_j)_q;\ x] = \sum_{k=0}^{\infty} \frac{\Gamma(a_1+e_1k)\,\Gamma(a_2+e_2k)\ \ldots\ \Gamma(a_p+e_pk)\,x^k}{\Gamma(b_1+f_1k)\,\Gamma(b_2+f_2k)\ \ldots\ \Gamma(b_q+f_q)\quad k!}$$

is the generalized hypergeometric function defined by WRIGHT ([**7**]$_1$, p. 287) and

(1.3) $$J_\nu^\mu(x) = \sum_{r=0}^{\infty} \frac{(-x)^r}{r!\,\Gamma(1+\nu+\mu r)}$$

is the generalized BESSEL function defined by WRIGHT ([**7**]$_2$, p. 257).

The functions (1.2) and (1.3) being special cases of the H function defined by FOX ([**2**], p. 408), their LAPLACE transforms can be obtained as special cases of the LAPLACE transform of the H function proved by GUPTA ([**3**], p. 100). The following two special cases will be used in this paper:

(1.4) $$t^\alpha\,{}_1\psi_1[(\delta, 1);\ (1+\alpha, b);\ -ct^b] \doteqdot \Gamma(\delta)\,p^{-1-\alpha}(1+cp^{-b})^{-\delta}\ ,$$

provided Re $(p) > 0$, $2 > b > 0$, Re $(1+\alpha) > 0$ and $|\arg cp^{-b}| < \pi(2-b)/2$;

(1.5) $$t^\alpha J_\alpha^\beta(at^\beta) \doteqdot p^{-1-\alpha}\exp[-ap^{-\beta}]\ ,$$

provided Re $(p) > 0$, $1 \geqslant \beta > 0$, Re $(1+\alpha) > 0$ and $|\arg ap^{-\beta}| < \pi(1-\beta)/2$. From ERDÉLYI ([**1**], pp. 129-131) we have:

(1.6) If $f(t) \doteqdot g(p)$, then $\exp[-at]f(t) \doteqdot g(p+a)$.

(1.7) If $f(t) \doteqdot g(p)$, $f(0) = f'(0) = \ldots = f^{(n-1)}(0) = 0$ and $f^{(n)}(t)$ is continuous, then $f^{(n)}(t) \doteqdot p^n g(p)$.

(1.8) If $f_1(t) \doteqdot g_1(p)$ and $f_2(t) \doteqdot g_2(p)$, then

$$\int_0^t f_1(x) f_2(t-x)\,\mathrm{d}x \doteqdot g_1(p) g_2(p)\,,$$

(1.9) $t^{\delta-1}\,{}_mF_n[{}_1(\alpha_j)_m;\ {}_1(\beta_j)_n;\ (at)]^k \doteqdot$

$$\doteqdot \Gamma(\delta)\,p^{-\delta}\,{}_{m+k}F_n[{}_1(\alpha_j)_m,\ \Delta(k,\delta);\ {}_1(\beta_j)_n;\ (ka/p)^k]\,,$$

provided $m+k \leqslant n+1$, $\operatorname{Re}(\delta) > 0$, $\operatorname{Re}(p) > 0$ when $m+k \leqslant n$ and $\operatorname{Re}[p + ka\cdot\exp[2\pi ir/k]] > 0$ for $r = 0, 1, \dots, k-1$ when $m+k = n+1$.

2. - The integral equations.

Theorem I. *Each of integral equations*

(2.1) $$g(t) = A\int_0^t [(D+a)^m f(t-x)] \exp[-ax]\, x^\alpha\, {}_1\psi_1[(\delta, 1);\ (1+\alpha, b);\ cx^b]\,\mathrm{d}x$$

and

(2.2) $$f(t) = B\int_0^t [(D+a)^n g(t-x)] \exp[-ax]\, x^\beta\, {}_1\psi_1[(-\delta, 1);\ (1+\beta, b);\ cx^b]\,\mathrm{d}x$$

is the solution of the other, provided

(2.3) $$\begin{cases} 2 > b > 0\,,\ \operatorname{Re}(1+\alpha) > 0\,,\ \operatorname{Re}(1+\beta) > 0\,,\ AB\Gamma(\delta)\Gamma(-\delta) = 1\,, \\ \alpha+\beta+2 = m+n, \end{cases}$$

(2.4) $f(0) = f'(0) = \dots = f^{(m-1)}(0) = 0$, $\quad f^{(m)}(x)$ *is continuous*,

(2.5) $g(0) = g'(0) = \dots = g^{(n-1)}(0) = 0$, $\quad g^{(n)}(x)$ *is continuous*,

(2.6) *m and n are non-negative integers and D represents differentiation with respect to $t-x$.*

Proof. Let $f(t) \doteqdot F(p)$ and $g(t) \doteqdot G(p)$. Using (1.4) and (1.6),

$$\exp[-at]\,t^\alpha\,{}_1\psi_1[(\delta, 1);\ (1+\alpha, b);\ ct^b] \doteqdot (p+a)^{-1-\alpha}[1 + c(p+a)^{-b}]^{-\delta}\,\Gamma(\delta)\,.$$

Using (1.7),

$$(D+a)^m f(t) \doteqdot F(p)(p+a)^m\,,$$

provided the conditions (2.4) are satisfied.

Using (1.8), the integral equation (2.1) becomes

$$G(p) = A\Gamma(\delta)(p+a)^{m-1-\alpha}[1-c(p+a)^{-b}]^{-\delta}F(p)\,. \tag{2.7}$$

Similarly, the integral equation (2.2) becomes

$$F(p) = B\Gamma(-\delta)(p+a)^{n-1-\beta}[1-c(p+a)^{-b}]^{\delta}G(p)\,. \tag{2.8}$$

The equations (2.7) and (2.8) can be obtained from each other when $AB\Gamma(-\delta)\Gamma(\delta)=1$ and $m+n=\alpha+\beta+2$. Hence, by LERCH's theorem ([**5**], p. 5), it follows that each of the integral equations (2.1) and (2.2) is the solution of the other.

Special cases. Putting $b = r/s$, where r and s are positive integers, the theorem (2.2) gives the following result involving MEIJER's G function.

Each of the integral equations

$$g(t) = A\int_0^t [(D+a)^m f(t-x)]\exp[-ax]x^{\alpha}\, G^{s,s}_{s,s+r}\left[cx^r \,\middle|\, \begin{matrix}\Delta(s, 1-\delta)\\ \Delta(s,0),\ \Delta(r,-\alpha)\end{matrix}\right] dx \tag{2.9}$$

and

$$f(t) = B\int_0^t [(D+a)^n g(t-x)]\exp[-ax]x^{\beta}\, G^{s,s}_{s,s+r}\left[cx^r \,\middle|\, \begin{matrix}\Delta(s, 1+\delta)\\ \Delta(s,0),\ \Delta(r,-\beta)\end{matrix}\right] dx \tag{2.10}$$

is the solution of the other, provided $\mathrm{Re}\,(1+\alpha)>0$, $\mathrm{Re}\,(1+\beta)>0$, $r<2s$, $\alpha+\beta+2=m+n$, the conditions (2.4), (2.5) and (2.6) of (2.2) are satisfied and

$$\Gamma(\delta)\Gamma(-\delta)AB = r^{1-m-n}(2\pi)^{1+r-2s}\,.$$

When $r=s=1$, the G functions in (2.10) reduce to confluent hypergeometric functions. Put further $A=1$, $a=m=0$ to get a theorem by JET WIMP ([**4**], p. 44).

Theorem II. *Each of the integral equations*

$$g(t) = A\int_0^t [(D+k)^m f(t-x)]\exp[-kx]x^{\alpha} J^b_{\alpha}(ax^b)\,dx \tag{2.11}$$

and

$$(2.12)\qquad f(t)=\frac{1}{A}\int_0^t [(D+k)^n g(t-x)]\exp[-kx]x^\beta J_\beta^b(-ax^b)\,dx$$

is the solution of the other, provided $m+n=\alpha+\beta+2$, $1\geqslant b>0$, $\mathrm{Re}\,(1+\alpha)>0$, $\mathrm{Re}\,(1+\beta)>0$ *and the conditions* (2.4), (2.5) *and* (2.6) *of* (2.2) *are satisfied.*

The proof of (2.12) is similar to that of (2.2), using (1.5).

Special cases. Putting $b=r/s$, where r and s are positive integers, the theorem (2.12) reduces to the following result involving MEIJER's G function.

Each of the integral equations

$$(2.13)\quad g(t)=A\int_0^t [(D+k)^m f(t-x)]\exp[-kx]x^\alpha G_{0,s+r}^{s,0}[ax^r\,|\,\Delta(s,0),\Delta(r,-\alpha)]\,dx$$

and

$$(2.14)\quad f(t)=B\int_0^t[(D+k)^n g(t-x)]\exp[-kx]x^\beta G_{0,s+r}^{s,0}[(-1)^s ax^r\,|\,\Delta(s,0),\Delta(r,-\beta)]\,dx$$

is the solution of the other, provided $r\leqslant s$, $\mathrm{Re}\,(1+\alpha)>0$, $\mathrm{Re}\,(1+\beta)>0$, $\alpha+\beta=m+n+2$, $ABr^{m+n+3}=s(2\pi)^{r-s}$ and the conditions (2.4), (2.5) and (2.6) of (2.2) are satisfied.

When $r=s=1$, the G functions of (2.14) reduce to BESSEL functions.

Various special cases of (2.14) were proved by RUSIA ([**6**]$_3$, pp. 67-70).

Theorem III. *If* k, m, n, r, s *are all integers,* $r>0$ $\mathrm{Re}\,(\alpha)>0$, $\mathrm{Re}\,(\beta)>0$, $r(n+s)+k+m=\alpha+\beta$,

$$f(0)=f'(0)=\ldots=f^{(i-1)}(0)=0=g(0)=g'(0)=\ldots=g^{(j-1)}(0)$$

and $f^{(i)}(t)$, $g^{(j)}(t)$ *are continuous functions, then each of the integral equations*

$$(2.15)\quad g(t)=\frac{1}{\Gamma(\alpha)}\int_0^t \{(D+b)^k[(D+b)^r+a^r]^s g(x)\}\exp[-b(t-x)](t-x)^{\alpha-1}\cdot$$

$$\cdot\,{}_1F_r[s+v;\,\Delta(r,\alpha);\,-(at-ax)^r/r^r]\,dx$$

and

$$(2.16)\quad g(t) = \frac{1}{\Gamma(\beta)} \int_0^t \{(D+b)^m[(D+b)^r + a^r]^n f(x)\} \exp[-b(t-x)](t-x)^{\beta-s} \cdot {}_1F_r[n-v;\ \Delta(r,\beta);\ -(at-ax)^r/r^r]\,\mathrm{d}x$$

is the solution of the other, where i is the number of times $f(x)$ *is differentiated in* (2.16), *j is the number of times* $g(x)$ *is differentiated in* (2.15),

$$D = \frac{\mathrm{d}}{\mathrm{d}x},\ \frac{1}{D} f(x) = \int_0^\infty f(x)\,\mathrm{d}x, \qquad \frac{1}{D+a} f(x) = \exp[-ax] \int_0^\infty \exp[ax] f(x)\,\mathrm{d}x.$$

The proof of (2.16) is similar to that of (2.2), using the following special case of (1.9):

$$t^{\alpha-1}{}_1F_n[v;\ \Delta(n.\ \alpha);\ -(at/n)^n] \doteqdot \Gamma(\alpha)\,p^{nv-\alpha}(p^n + a^n)^{-v},$$

provided $\mathrm{Re}(\alpha) > 0$ and $\mathrm{Re}(p+a) > 0$.

Special cases. Put $r=1$, $b=k=s=0$ to get a theorem by JET WIMP ([4], p. 44).

Various results of RUSIA ($[\mathbf{6}]_1$, pp. 119, 123, 127, 129; $[\mathbf{6}]_2$, pp. 16, 17) can be deduced from (2.16).

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Summary.

See the Introduction.

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