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CAROL SINGH (*)

On a class of integral equations. (**)

1. - Introduction.

In recent years a number of integral equation belonging mainly to the following two classes have been solved ([**1**]$_1$, [**1**]$_2$, [**2**]$_1$, [**6**], [**7**]). They are

(1.1) $$\int_x^1 K(x/t)\, g(t)\, \mathrm{d}t = f(x)$$

and

(1.2) $$\int_0^x K(x-t)\, g(t)\, \mathrm{d}t = f(x)$$

$K(x)$ is the kernel.

TA LI [**6**] at first solved an integral equation of the class (1.1) which he encountered in a certain aerodynamical problem. WIDDER [**7**] applied the methods of operational calculus to invert the convolution transforms (1.2).

A. ERDELYI [**2**]$_2$ recently gave the solution of the integral equation

(1.3) $$\int_x^\infty (t^2-x^2)^{\lambda/2} P_\nu^{-\lambda}(x/t)\, g(t)\, \mathrm{d}t = f(x)\,,$$

where $P_\nu^{-\lambda}(x)$ is a LEGENDRE function. The above integral equation belongs to the class of integral equations

(1.4) $$\int_x^\infty K(x/t)\, g(t)\, \mathrm{d}t = f(x)\,,$$

(*) Indirizzo: Department of Mathematics, Govt. Polytechnic Institute, Raigarh (M.P.), India.

(**) Ricevuto: 22-V-1970.

which is different from (1.1) in the respect that it possesses infinity as the constant limit of integration. A similar problem was attempted by SRIVASTAVA [5]. Obviously in the same lines considering the class (1.2) yet another class of integral equations can be defined:

$$\int_x^\infty K(t-x)\, g(t)\, \mathrm{d}t = f(x)\,. \tag{1.5}$$

We propose in this paper to solve one of the integral equation of the class (1.5), i.e.:

$$\int_x^\infty (t-x)^{\mu-\frac{1}{2}} M_{k,\mu}(x-t)\, g(t)\, \mathrm{d}t = f(x)\,, \tag{1.6}$$

where $M_{k,\mu}(x)$ is a WHITTAKER's function. We want to mention that in [4] we solved the integral equation

$$\int_0^x (x-t)^{\mu-\frac{1}{2}} M_{k+\frac{1}{2},\mu}\left\{2(x-t)\right\} g(t)\, \mathrm{d}t = f(x)\,. \tag{1.7}$$

2. - Notations and formulae.

The functions $M_{k,\mu}(x)$ and $W_{k,\mu}(x)$ are the WHITTAKER's functions they are defined as

$$M_{k,\mu}(x) = x^{\mu+\frac{1}{2}} \exp\,[-x/2]\, {}_1F_1(\tfrac{1}{2}+\mu-k;\, 2\mu+1;\, x)\,,$$

$$W_{k,\mu}(x) = \frac{\Gamma((-2\mu)\; M_{k,\mu}(x))}{\Gamma(\frac{1}{2}-\mu-k)} + \frac{\Gamma((2\mu)\; M_{k,-\mu}(x))}{\Gamma(\frac{1}{2}+\mu-k)}\,.$$

The LAPLACE transform of a function $f(x)$ is defined as

$$\int_0^\infty \exp\,[-px]\, f(x)\, \mathrm{d}x = F(p)\,.$$

We shall denote it symbolically as

$$Lf(x) = F(p)\,.$$

The inverse LAPLACE transform of the function is defined as

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \exp[px] F(p) \, dp$$

and we shall denote it as

$$f(x) = L^{-1} F(p) .$$

STIELTJES transforms are iterated LAPLACE transforms. They are defined as

$$g(y) = \int_0^\infty f(x)(x+y)^{-1} \, dx ,$$

or

$$g(y) = \int_0^\infty \exp[-ty] \left\{ \int_0^\infty \exp[-xt] f(x) \, dx \right\} dt,$$

i.e.

$$g(y) = L^2 f(x) .$$

From ([2]$_3$, p. 210-215) we have

$$L\left\{t^{\mu-\frac{1}{2}} M_{k,\mu}(at)\right\} = a^{\mu+\frac{1}{2}} \Gamma(2\mu+1) \frac{(p-a/2)^{k-\mu-\frac{1}{2}}}{(p+a/2)^{k+\mu+\frac{1}{2}}} . \tag{2.1}$$

Again from ([2]$_3$, pp. 293-294)

$$L^{-1}\left\{p^{-\mu-\frac{1}{2}} W_{k,\mu}(p)\right\} = \frac{1}{\Gamma(\mu-k+\frac{1}{2})} \frac{(t-\frac{1}{2})^{\mu-k-\frac{1}{2}}}{(t+\frac{1}{2})^{-\mu-k+\frac{1}{2}}} , \qquad t > \tfrac{1}{2} , \qquad \mu - k > \tfrac{1}{2} \tag{2.2}$$

and from ([2]$_3$, pp. 121-131) we have

$$L\left\{f_1(x) \cdot f_2(x)\right\} = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} g_1(z) \, g_2(p-z) \, dz , \tag{2.3}$$

provided

$$L f_1(x) = g_1(p) \qquad \text{and} \qquad L f_2(x) = g_2(p) .$$

We shall use the following theorem given by GUPTA [3]. If

$$Lf(x) = \psi(p) ,$$

$$Lh(x) = g(p)$$

and

$$L\{f(x)\cdot g(x)\} = \varphi(p) = L\{L^{-1}\psi(p)\cdot Lh(x)\} ,$$

then

$$\int_0^\infty \psi(x)\, h(x-p)\, \mathrm{d}x = \varphi(p), \tag{2.4}$$

provided the LAPLACE transform of $|f(t)|$, $|h(t)|$, and $|g(t)\cdot f(t)|$ exist and the integral (2.4) is absolutely convergent.

3. – Theorem. *If the function $f(x)$ exists then the integral equation* (1.6) *will have its solution $g(x)$ represented in any one of the following three forms:*

$$g(x) = \frac{A}{2\pi i}\int_{c-2\infty}^{c+i\infty} p^{-\mu-3/2}\, W_{-k,\,\mu+1}(p)\, f(p-x)\, \mathrm{d}p \tag{3.1}$$

or

$$g(x) = A \int_x^\infty p^{-\mu-3/2}\, W_{-k,\,\mu+1}(p)\, f_1(p-x)\, \mathrm{d}p \tag{3.2}$$

or

$$g(x) = B \int_x^\infty (t-x)^{-\mu-3/2}\, M_{k,\,-(\mu+1)}(t-x)\, f(t)\, \mathrm{d}t \ , \tag{3.3}$$

provided all the integral (3.1), (3.2) *and* (3.3) *are absolutely converyent.*

Here

$$A = \frac{\Gamma(\mu+k+\frac{3}{2})}{(-)^{\mu+1/2}\,\Gamma(2\mu+1)}$$

$$B = \left\{(-)^{\mu+1/2}\,\Gamma(2\mu+1)\,\Gamma(-2\mu-1)\right\}^{-1}$$

and $f(x)$ *is the* S t i e l t j e s *transformation of the function* $f_1(t)$, i.e.

$$L^2\{f_1(t)\} = f(x) .$$

4. – P r o o f. Taking the inverse LAPLACE transformation of the integral equation (1.6) in the light of result (2.4), we get

$$[L\{t^{\mu-1/2} M_{\mu,k}(-t)\}]\,[\{L^{-1} g(p)\}] = L^{-1} f(p) .$$

Applying the known result (2.1) we get

$$L^{-1} g(p) \frac{(-)^{\mu+1/2}\Gamma\big((2\mu+1)(x+\frac{1}{2})^{k-\mu-1/2}\big)}{(x-\frac{1}{2})^{k+\mu+1/2}} = L^{-1} f(p) \tag{4.1}$$

this gives

$$L^{-1} g(p) = \frac{1}{(-)^{\mu+1/2}\Gamma(2\mu+1)} \left\{\frac{(x-\frac{1}{2})^{k+\mu+1/2}}{(x+\frac{1}{2})^{k-\mu-1/2}}\right\} L^{-1} f(p) . \tag{4.2}$$

But applying (2.2) this can be re-written as

$$L^{-1} g(p) = \frac{\Gamma(\mu+k+\frac{3}{2})}{(-)^{\mu+1/2}\Gamma(2\mu+1)} \left\{L^{-1}(p^{-\mu-3/2} W_{-k,\mu+1}(p))\right\} L^{-1} f(p) \tag{4.3}$$

or

$$L^{-1} g(p) = \frac{\Gamma(\mu+k+\frac{3}{2})}{(-)^{\mu+1/2}\Gamma(2\mu+1)} \left\{L^{-1}(p^{-\mu-3/2} W_{-k,\mu+1}(p))\right\} \left\{L(L^{-2} f(p))\right\} . \tag{4.4}$$

(3.2) can also be rewritten as

$$L^{-1} g(p) = \frac{1}{(-)^{k-1/2}\,\Gamma(2\mu+1)\,\Gamma(-2\mu-1)} \left\{L(M_{k,(-k+1)}(t))\right\} \left\{L^{-1} f(p)\right\} . \tag{4.5}$$

(*a*) Applying the result (2.3) in taking the LAPLACE transform of (4.3) we get the result (3.1).

(*b*) Applying the result (2.4) while taking the LAPLACE transformation of (4.4) we get (3.2).

(*c*) Lastly the result (2.4) when applied in taking the LAPLACE transform of (4.5) we get (3.3).

This completes the proof.

5. - Evaluation of a few integrals.

Let us evaluate the integral

$$(5.1)\qquad \int_0^\infty t^{\nu-1/2} M_{\mu,\nu}(-t)(t+x)^{-\mu-1/2} W_{k,\mu}(t+x)\,\mathrm{d}t = I(x)\,,$$

which can be written as

$$(5.2)\qquad \int_x^\infty (t-x)^{\nu-1/2} M_{\lambda,\nu}(x-t)\, t^{-\mu-1/2} W_{k,\mu}(t)\,\mathrm{d}t = I(x)\,.$$

Applying GUPTA's theorem (2.4) in view of the results (2.1) and (2.2) on the integral (5.2) we get

$$\frac{(-)^{\nu+1/2}\Gamma(2\nu+1)}{\Gamma(\mu-k+\frac{1}{2})}\,\frac{(z+\frac{1}{2})^{\lambda-\nu-1/2}}{(z-\frac{1}{2})^{\lambda+\nu+1/2}}\,\frac{(z-\frac{1}{2})^{\mu-k-1/2}}{(z+\frac{1}{2})^{-\mu-k+1/2}} = L^{-1}I(x);$$

this gives

$$(5.3)\qquad L^{-1}I(x) = \frac{(-)^{\nu+1/2}\,\Gamma(2\nu+1)}{\Gamma(\mu-k+\frac{1}{2})}\,\frac{(z-\frac{1}{2})^{(\mu-\nu-1/2)-(\lambda+k)-1/2}}{(z+\frac{1}{2})^{-(\mu-\nu-1/2)-(\mu+k)+1/2}}\,,$$

$$(5.4)\qquad I(x) = \frac{(-)^{\nu+1/2}\Gamma(2\nu+1)\,\Gamma(\mu-\nu-\lambda-k)}{\Gamma(\mu-k+\frac{1}{2})}\,x^{\nu-\mu}\,W_{k+\mu,\mu-\nu-1/2}(x)\,.$$

If we keep $\nu=\lambda-\frac{1}{2}$ in the integral (5.1) we get the known result ([**3**]$_4$, pp. 411-450).

Inverting (5.2) in the light of the inversion formulae (3.1) and (3.3) we evaluate two more integrals

$$(5.5)\qquad t^{-\mu-1/2}W_{k,\mu}(t) = A\int_{c-i\infty}^{c+i\infty} x^{-\nu-3/2}\,W_{-\lambda,\,\nu+1}(x)(x-t)^{\nu-\mu}\,W_{k+\lambda,\,\mu-\nu-1/2}(x-t)\,\mathrm{d}x\,,$$

where

$$A = \frac{\Gamma(\lambda+\nu+\frac{3}{2})\,\Gamma(\mu-\nu-\lambda-k)}{2\pi i\,\Gamma(\mu-k+\frac{1}{2})}\,,$$

and

$$(5.6)\qquad t^{-\mu-1/2}W_{k,\mu}(t) = B\int_x^\infty (x-t)^{-\nu-3/2} M_{\lambda,\,-(\nu+1)}(x-t)\,x^{\nu-\mu}\,W_{k+\lambda,\,\mu-\nu-1/2}(x)\,\mathrm{d}x\,,$$

where

$$B = \frac{\Gamma(\mu - \nu - \lambda - k)}{\Gamma(\mu - k + \frac{1}{2})\,\Gamma(-2\nu - 1)}\,.$$

The above results can be written in a little better form

$$(5.7)\qquad t^{-s-2}\, W_{r+p,\, s+2-\frac{1}{2}}(t) = A_1 \int_{c-i\infty}^{c+i\infty} x^{-2-\frac{1}{2}}\, W_{p,2}(x)(x-t)^{-s-\frac{1}{2}}\, W_{r,s}(x-t)\, dx\,,$$

$$(5.8)\qquad t^{2-s}\, W_{r-p,\, s-2-\frac{1}{2}}(t) = B_1 \int_{x}^{\infty} (x-t)^{2-\frac{1}{2}}\, M_{p,2}(x-t)\, x^{-s-\frac{1}{2}}\, W_{r,s}(x)\, dx\,,$$

where

$$A_1 = \frac{\Gamma(2-p+\frac{1}{2})\,\Gamma(s-r+\frac{1}{2})}{2\pi i\,\Gamma(s+2-r-p)}\,, \qquad B_1 = \frac{\Gamma(s-r+\frac{1}{2})}{\Gamma(s-2-r+p)\,\Gamma(2q+1)}\,.$$

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