Riv. Mat. Univ. Parma (2) 3 (1974), 21-32

R. Ascoli, G. Epifanio, A. Restivo (*)

*-Algebrae of unbounded Operators in Scalar-Product Spaces. (**)

ad Antonio Mambriani per il suo 75-mo compleanno

1. - Introduction.

In one previous work [1] we investigated a mathematical formalism to describe the quantum fields at a point. This research led us to consider some particularly simple algebrae of linear operators in non-complete scalar-product spaces.

Precisely, we were led to introduce the following definition.

Definition 1. *Let D be a scalar product space. We say that a linear operator A defined on D has an adjoint A^* in D whenever there exists a linear operator A^* defined on D such that*

$$\forall\, \varphi, \psi \in D \qquad (A\varphi \,|\, \psi) = (\varphi \,|\, A^* \psi)\,.$$

We call C_D the set of the linear operators that are defined on D and have an adjoint in D; we call B_D the subset of C_D consisting of the bounded operators.

We endow C_D (and B_D) with the D-weak topology, determined by the set of seminorms

$$\{A \to |\,(A\varphi \,|\, \psi)\,|\, \big| \varphi, \psi \in D\}\,.$$

(*) Indirizzo degli Autori: Istituto di Fisica dell'Università di Lecce, ora allo Istituto di Matematica dell'Università di Parma; Istituto di Fisica dell'Università di Palermo; Laboratorio di Cibernetica del C.N.R., Arco Felice, Napoli.

(**) This work has been performed in part within the I.N.F.N. and in part within the Gruppo Nazionale per la Fisica Matematica del C.N.R. — Ricevuto: 1-II-1973.

Then we remarked that the WIGHTMAN quantum fields [1] may be defined as C_D-valued distributions on the space-time satisfying some axioms and we proved that actually, as a consequence of some of the axioms, such fields « are » functions on the space-time with values in the sequential completion $\tilde{C}_D$ of C_D (which in general is no more a space of operators).

So we were led to consider, besides B_D and C_D, their sequential completions $\tilde{B}_D$ and $\tilde{C}_D$.

Let us further introduce the definition:

Definition 2. *Let D be a scalar product space. We call D_W the space D endowed with the D-weak topology, determined by the set of seminorms: $\{\varphi \to |(\varphi|\psi)| \,|\, \psi \in D\}$. We call $\mathscr{L}(D_W)$ the space of the linear continuous operators of D_W, endowed with the D-weak topology (see Definition 1).*

Whenever D is complete, the spaces B_D, C_D, $\tilde{B}_D$, $\tilde{C}_D$, $\mathscr{L}(D_W)$ coincide, because:

$B_D = \mathscr{L}(D_W)$	as a known consequence of the RIESZ theorem together with the uniform boundedness theorem
$C_D = B_D$	due to the closed graph theorem
$\tilde{B}_D = B_D$	due to the uniform boundedness theorem.

These spaces do actually coincide, for complete D, with the well known space $\mathscr{L}(H)$ of the bounded operators of the HILBERT space, endowed with the weak topology.

If D is not complete, in general the above equalities do not hold [2].

In this paper we study some easy properties of the above mentioned spaces, and also of the completions $\hat{B}_D$ and $\hat{C}_D$. In particular we find that $C_D = \mathscr{L}(D_W)$, $\hat{B}_D = \hat{C}_D$ and $\tilde{B}_D = \tilde{C}_D$, this latter statement being true under the assumption of separable D.

The fact is important, that B_D and C_D have a structure of topological $*$-algebrae [3]. With this structure C_D appears, from some points of view, as the most natural algebra of unbounded operators which reduces to $\mathscr{L}(H)$ when D is complete [4]. Moreover, for any HILBERT space H and for any

---

(1) See for instance Ref. [**10**].

(2) In the appendix we give a simple example of an element of $\tilde{B}_D$ which does not represent any operator.

(3) The algebra C_D has also meanwhile been considered by G. LASSNER [**2**]. We thank the author for having sent us his paper before publication.

(4) See also Ref. [**7**] page 88.

$*$-algebra $\mathscr{A}$ of linear operators of H in which the sum and the product are defined as it is usually done for unbounded operators, there exists a dense linear manifold D of H such that $\mathscr{A}$ is a $*$-subalgebra of C_D.

2. - The topological $*$-algebrae B_D and C_D.

With reference to Definition 1 we have firstly:

Theorem 1. *The elements of C_D are closed operators in D* (5).

In fact in any scalar product space the adjoint of every operator with a dense domain is a closed operator (6). According to Definition 1 any operator A of C_D is the adjoint of the operator A^*, hence it is closed.

In our work [**1**] we had already remarked the following proposition, whose proof is straightforward.

Theorem 2. *For any scalar product space D, C_D (endowed with the natural operations) is a topological $*$-algebra in the following sense:*

a) *C_D is an algebra with involution*
b) *C_D is a locally convex Hausdorff-space*
c) *the product is separately continuous*
d) *the involution is continuous.*

B_D is a topological $*$-subalgebra of C_D.

The algebrae C_D are of interest with reference to a remarkably general type of algebrae of unbounded operators of the Hilbert space. In this connection we recall that working with sets of unbounded operators usually requires much care concerning the domains of definition. In particular when dealing with algebrae of operators, generalizations of the definitions of sum and product sometimes are required (7). However, if we preserve the usual meaning for the operations of sum and product of operators, the following statement holds.

---

(5) Of course C_D does not generally contain every closed linear operator in D. as well as B_D does not generally contain every bounded linear operator in D.

(6) The usual proof (see for instance Ref. [**5**] Ch. I § **5** n° 9) does not use the completeness of the space.

(7) See for instance Ref. [**6**] Ch. IX § **128**.

Theorem 3. *Let H be an Hilbert space and let $\mathscr{A}$ be any $*$-algebra of (not necessarily bounded) densely definite linear operators of H, such that the sum and the product are defined as it is usually done for unbounded operators and such that the $*$-operation transforms any operator of $\mathscr{A}$ into a restriction of its adjoint. Then there exists a dense linear manyfold D of H such that $\mathscr{A}$ is a $*$-subalgebra of C_D.*

Indeed the requirement that addition has the group properties implies firstly that all the operators of $\mathscr{A}$ have the same domain: in fact for any A, $B \in \mathscr{A}$ we have (D_A meaning the domain of A, etc.)

$$A + B - B = A \Rightarrow D_{A+B} \cap D_B = D_A \Rightarrow D_A \subset D_B .$$

Analogously it may be obtained $D_B \subset D_A$, so that finally for any $A, B \in \mathscr{A}$ $D_A = D_B$. Furthermore we have to require that the product of any two operators of $\mathscr{A}$ be an operator of $\mathscr{A}$, hence that its domain be D: it follows immediately that all the operators of $\mathscr{A}$ must leave D invariant. As a consequence it turns out finally that the $*$-operation within $\mathscr{A}$ coincides with the adjonction in D, that has been introduced in Definition 1. So, according to Definition 1, $\mathscr{A}$ is a $*$-subalgebra of C_D.

Hence it is just C_D that for unbounded $*$-representations of $*$-algebrae takes most naturally the place $\mathscr{L}(H)$ holds for bounded $*$-representations. Examples of such unbounded $*$-representations are met in the theory of the representations of LIE algebrae as well as in the already mentioned WIGHTMAN formulation of quantum field theory (4).

With reference to Definition 2 we have

Theorem 4. *For any scalar product space D it is* (8) $C_D = \mathscr{L}(D_W)$ *and the equality is an isomorphism of topological $*$-algebrae (with separately continuous product), with respect to the natural operations.*

Proof. Let us call $\underline{D}$ the scalar product space that is the complex conjugate of D (9). Let us then consider the spaces D and $\underline{D}$ as forming a pairing (10) with respect to the bilinear functional $(\varphi, \psi) \to (\varphi | \psi)$ that is defined on $D \times \underline{D}$ by the scalar product of D. This pairing separates points of both D and $\underline{D}$, so that $(D, \underline{D})$ is a dual system with respect to such a bilinear form. Let further A be any linear operator of D (defined on the whole D). Then a

---

(8) Hence any element of C_D may be extended by continuity to a linear operator in the sequential completion $\tilde{D}_W$ or even in the completion $\hat{D}_W$.

(9) See Ref. [9], Ch. V, § 1, n° 3.

(10) See Ref. [4], Ch. II, § 6, n° 1.

known proposition [11] states the equivalence of the following two statements:

1) A is continuous with respect to the weak topology [12] $\sigma(D, \underline{D})$ defined on D by the pairing under consideration.

2) There exists an operator A^* of $\underline{D}$ (defined on the whole $\underline{D}$) such that

$$\forall \varphi \in D \qquad \forall \psi \in \underline{D} \qquad (A\varphi | \psi) = (\varphi | A^* \psi) .$$

Then we remark that the set supporting $\underline{D}$ is D and, for any operator of D, linearity with respect to the original structure of D is equivalent to linearity with respect to the structure of $\underline{D}$. We remark further that the $\sigma(D, \underline{D})$ weak topology on D coincides with the D-weak topology of Definition 2.

Then the statement of the theorem follows from the definitions of C_D and $\mathscr{L}(D_W)$.

We remark that the statement *a*) of Theorem 2 does also immediately follow from Theorem 4.

3. - The Completions $\hat{B}_D$ and $\hat{C}_D$.

In order to compare the completions $\hat{B}_D$ and $\hat{C}_D$, it is convenient to compare both of them with the algebraic dual $(D \otimes \underline{D})^*$ of the algebraic tensor product of D and its complex conjugate $\underline{D}$.

To this purpose we use the following proposition, which collects known facts concerning weak topologies [13].

Lemma 1. *Let two vector spaces X and Y form a pairing with respect to some bilinear functional $(x, y) \to B(x, y)$, which is supposed to separate points of both X and Y. Let us endow X with the weak topology $\sigma(X, Y)$; let us call Y^* the algebraic dual of Y and let us endow it with the weak topology $\sigma(Y^*, Y)$.*

Then the canonical mapping of X into Y^*, which transforms any element x of X into the element of Y^* $y \to B(x, y)$, is injective, linear and bicontinuous. Its continuous extension to the completion $\hat{X}$ of X is an isomorphism of the topological vector spaces $\hat{X}$ and Y^*.

Using this Lemma we prove the following proposition.

Lemma 2. *Let D be a scalar product space. Let us call $\underline{D}$ its complex conjugate* [9] *and $(D \otimes \underline{D})^*$ the algebraic dual of the algebraic tensor product*

(11) See Ref. [4], Ch. II, § **6**, n° 4, Prop. 5 and Cor. .
(12) See Ref. [4], Ch. II, § **6**, n° 2, Def. 2.
(13) See Ref. [4], Chap. II, § **6**, n° 1 and § **6**, n° 9, Prop. 9.

$D \otimes \underline{D}$, endowed with the weak topology (¹²) *$\sigma((D \otimes \underline{D})^*, D \otimes \underline{D})$. Then the linear mapping $A \to \overset{\otimes}{F}_A$ of C_D into $(D \otimes \underline{D})^*$ defined by* (¹⁴)

$$\forall\, \varphi, \psi \in D \qquad \overset{\otimes}{F}_A(\varphi \otimes \psi) = (A\varphi \,|\, \psi)$$

is injective and bicontinuous. Its continuous extension to $\hat{C}_D$ (or $\hat{B}_D$) provides an isomorphism of the topological vector spaces $\hat{C}_D$ (or $\hat{B}_D$) and $(D \otimes \underline{D})^*$.

Proof. We consider the spaces C_D (or B_D) and $D \otimes \underline{D}$ as forming a pairing [10] with respect to the bilinear functional $(A, y) \to B(A, y)$ defined by

$$\forall\, \varphi, \psi \in D \qquad \forall\, A \in C_D \qquad B(A, \varphi \otimes \psi) = (A\varphi \,|\, \psi)\,.$$

This pairing separates points of both C_D (or B_D) and $D \otimes \underline{D}$. If we in fact assume $\forall\, y \in D \otimes \underline{D}$ $B(A, y) = 0$, then we have in particular $\forall\, \varphi, \psi \in D$ $(A\varphi \,|\, \psi) = 0$, hence $A = 0$, so that the pairing separates points of C_D (or B_D). It separates also points of $D \otimes \underline{D}$. Let in fact be $y = \sum_1^n{}_i\, \varphi_i \otimes \psi_i$, the φ_i's and ψ_i's belonging to D and the φ_i's being linearly independent. Let us assume $\forall\, A \in B_D$ $B(A, y) = 0$, that is $\forall\, A \in B_D$ $\sum_1^n{}_i\, (A\varphi_i \,|\, \psi_i) = 0$. Let us take for any $j = 1, 2 \dots n$ $A_j \in B_D$ such that for any $i = 1, 2 \dots n$ $A_j \varphi_i = \delta_{ij} \psi_i$ (A_j may for instance be choosen as being zero on the orthogonal complement of the set of the φ_i's). Then the assumption, when applied for any $j = 1, 2 \dots n$ to A_j, implies $(\psi_j \,|\, \psi_j) = 0$, hence $\psi_j = 0$. So it implies $y = 0$. Hence the pairing separates points of $D \otimes \underline{D}$.

Then we apply Lemma 1, with $X = C_D$ (or B_D) and $Y = D \otimes \underline{D}$.

The weak topology [12] $\sigma(C_D, D \otimes \underline{D})$ on C_D is determined by the set of seminorms

$$\{A \to |B(A, y)| \,|\, y \in D \otimes \underline{D}\}\,;$$

it is easily seen that this set of seminorms is equivalent to the set of seminorms

$$\{A \to |B(A, \varphi \otimes \psi)| \,|\, \varphi, \psi \in D\} = \{A \to |(A\varphi \,|\, \psi)| \,|\, \varphi, \psi \in D\}\,,$$

which defines on C_D the D-weak topology of Definition 1, so that the weak topology $\sigma(C_D, D \otimes \underline{D})$ on C_D coincides with the D-weak topology. An analogous statement holds for the space B_D.

(¹⁴) We recall that the set supporting $\underline{D}$ is D.

$\otimes$

The canonical mapping of C_D into $(D \otimes \underline{D})^*$ is the linear mapping $A \to F_A$ that is defined in the statement to be proved (and its restriction to B_D provides the canonical mapping of B_D into $(D \otimes \underline{D})^*$).

Then Lemma 1 states that this mapping is injective and bicontinuous and its continuous extension to $\hat{C}_D$ (or $\hat{B}_D$) provides an isomorphism of the topological vector spaces $\hat{C}_D$ (or $\hat{B}_D$) and $(D \otimes \underline{D})^*$.

So the Lemma is proved and the following proposition derives immediately.

Theorem 5. *For any scalar product space D it is, for the completions,* $\hat{B}_D = \hat{C}_D$.

We consider another easy consequence of Lemma 2. Let us describe any operator A of C_D by means of the sesquilinear form F_A on D $(\varphi, \psi) \to \to F_A(\varphi, \psi) = (A\varphi \mid \psi)$. Then the next proposition states that even every element of the completion $\hat{C}_D$ of C_D may correspondingly be described by means of a sesquilinear form; moreover all the sesquilinear forms on D are obtained in this way.

Theorem 6. *Let D be a scalar product space. Let us call S_D the space of the sesquilinear forms on D, endowed with the topology of pointwise convergence, which is determined by the set of seminorms*

$$\{F \to |F(\varphi, \psi)| \mid \varphi, \psi \in D\}.$$

Then the mapping from C_D (or B_D) into S_D defined for any $A \in C_D$ by $A \to F_A$, where

$$\forall \varphi, \psi \in D \qquad F_A(\varphi, \psi) = (A\varphi \mid \psi)$$

is injective, linear and bicontinuous. Its continuous extension to $\hat{C}_D$ provides an isomorphism of the topological vector spaces $\hat{C}_D$ and S_D [15].

---

[15] We remark that Theorem 6 is in accord with the fact that, even in the case of complete D, B_D, which is then the space $\mathscr{L}(H)$ of the bounded operators of the HILBERT space H, endowed with the weak topology, is not complete, though it is sequentially complete. In fact the elements of $\mathscr{L}(H)$ are known to correspond to the continuous sesquilinear forms on H. If $\mathscr{L}(H)$, endowed with the weak topology, would be complete, then, according to Theorem 6, the set of the continuous sesquilinear forms on H would coincide with the set S_H of all the sesquilinear forms on H, which is absurd.

The statement derives from Lemma 2 by considering the fact that the mapping $F \to \overset{\otimes}{F}$ of S_D into $(D \otimes \underline{D})^*$ defined by

$$\forall \varphi, \psi \in D \qquad \overset{\otimes}{F}(\varphi \otimes \psi) = F(\varphi, \psi)$$

is an isomorphism of topological vector spaces.

Indeed any sesquilinear form F on D, may be considered as a bilinear functional on $D \times \underline{D}$ and vice versa. Moreover $\overset{\otimes}{F}$ is just the linear functional on $D \otimes \underline{D}$ that corresponds to the bilinear functional F on $D \times \underline{D}$, according to the universal property of the algebraic tensor product. It follows immediately that the mapping $F \to \overset{\otimes}{F}$ is an algebraic isomorphism of S_D onto $(D \otimes \underline{D})^*$.

Concerning the topologies the mapping $F \to \overset{\otimes}{F}$ transforms the set of seminorms

$$\{F \to |F(\varphi, \psi)| \,|\, \varphi, \psi \in D\},$$

which defines the topology on S_D, in the set of seminorms

$$\{\overset{\otimes}{F} \to |\overset{\otimes}{F}(\varphi \otimes \psi)| \,|\, \varphi, \psi \in D\}$$

on $(D \otimes \underline{D})^*$. It is easily seen that this latter set of seminorms is equivalent to the set of seminorms

$$\{\overset{\otimes}{F} \to |\overset{\otimes}{F}(y)| \,|\, y \in D \otimes \underline{D}\}$$

which defines the weak topology $\sigma\big((D \otimes \underline{D})^*, D \otimes \underline{D}\big)$ on $(D \otimes \underline{D})^*$. So the mapping $F \to \overset{\otimes}{F}$ is also an homeomorphism of S_D onto $(D \otimes \underline{D})^*$ and the statement is proved ([16]).

([16]) Analogously let $L_W(D, \underline{D}^*)$ be the space of the linear operators from D into the algebraic dual $\underline{D}^*$ of the complex conjugate $\underline{D}$ of D, endowed with the weak operator topology. Then the canonical injection of C_D (or B_D) into $L_W(D, \underline{D}^*)$ extends continuously to an isomorphism of $\hat{C}_D$ and $L_W(D, \underline{D}^*)$.

Moreover let us endow $\underline{D}^*$ with the weak topology. Then we remark that, according to Lemma 1, the canonical injection $\varphi \to \varphi'$ of D_W into $\underline{D}^*$, defined by $\forall \psi \in D$ $\langle \varphi', \psi \rangle = (\varphi | \psi)$, is bicontinuous and its continuous extension to $\hat{D}_W$ provides an isomorphism of the topological vector spaces $\hat{D}_W$ and $\underline{D}^*$. It may easily be deduced that it is also canonically $\hat{C}_D = L_W(D, \hat{D}_W)$.

4. - The sequential completions $\tilde{B}_D$ and $\tilde{C}_D$ for separable D.

The propositions of this paragraph hold under the assumption of separable D.

In order to derive that, for sequential completions [17], it is $\tilde{B}_D = \tilde{C}_D$, we want to prove that $C_D \subset \tilde{B}_D$. Actually a somewhat stronger statement holds, in which the topology that is assumed on B_D is finer, according to the following definition.

Definition 3. *For any scalar product space D we call D-strong topology on C_D (or B_D) the topology that is defined by the set of seminorms*

$$\{A \to |A\varphi| \,|\, \varphi \in D\}.$$

Theorem 7. *For any separable scalar product space D, C_D is contained in the sequential completion of B_D endowed with the D-strong topology.*

Proof. Let (e_ν) be an orthonormal basis in D and let P_ν be the orthogonal projector on e_ν. Then, for any element A of C_D and for any ν, the product $P_\nu A$ is bounded because it is [18]

$$\forall \varphi \in D \qquad |P_\nu A\varphi| = |(A\varphi|e_\nu)| = |(\varphi|A^* e_\nu)| \leqslant |A^* e_\nu|\,|\varphi| \,.$$

So for any ν it is $P_\nu A \in B_D$.

Let us then consider the sequence (A_n) of operators of B_D, with $A_n = \sum_{1}^{n}{}_\nu P_\nu A$. As (e_ν) is an orthonormal basis, we have:

$$\forall \psi \in D \qquad \lim_{n \to \infty} \sum_{1}^{n}{}_\nu \; P_\nu \psi = \psi \,.$$

---

[17] The sequential completion $\tilde{X}$ of any HAUSDORFF topological vector space X is defined as the smallest sequentially complete subspace of the completion $\hat{X}$, which contains X. We recall that in general, whenever X does not satisfy the first countability axiom, $\tilde{X}$ does neither coincide with the set of the classes of equivalent CAUCHY sequences of X, nor with the completion $\hat{X}$.

[18] The boundedness of $P_\nu A$ is also a particular case of the proposition: in a normed space (and actually in more general spaces) any closable operator with finite rank is continuous. (For the case of normed spaces see **[8]**, page 166, Problem 5.18).

Using this statement with $\psi = A\varphi$, we get

$$\forall\, \varphi \in D \qquad \lim_{n\to\infty} A_n \varphi = \lim_{n\to\infty} \sum_{1}^{n}{}_{\nu} P_\nu A\varphi = A\varphi \, .$$

So the arbitrary element A of C_D, turns out to be the limit of the sequence (A_n) of elements of B_D with respect to the D-strong topology of C_D introduced in Definition 3. So the theorem is proved.

It follows immediately:

Theorem 8. *For any separable scalar product space D it is, for the sequential completions,* $\tilde{B}_D = \tilde{C}_D$.

In fact, the D-weak topology being coarser than the D-strong topology, the statement of Theorem 7 holds also with the D-weak topology on B_D, that is $C_D \subset \tilde{B}_D$. It follows $\tilde{C}_D \subset \tilde{B}_D$. It is obviously $\tilde{B}_D \subset \tilde{C}_D$, hence it is $\tilde{B}_D = \tilde{C}_D$.

Appendix.

Let us give a simple example to show that in general B_D and C_D are not sequentially complete and there exist elements of their sequential completion which do not represent operators of D (nor operators from D into $H = \hat{D}$) (19).

Let D be the space of the (finite) linear combinations of the vectors of an orthonormal basis (e_ν) of an infinite-dimensional separable HILBERT space H. Clearly, in this case, any sesquilinear form F on D (that is any element of S_D) may be represented by means of the matrix $(F_{\mu\nu}) = (F(e_\mu, e_\nu))$ and conversely any (countably infinite) matrix represents an element of S_D. Moreover the pointweise convergence in S_D corresponds to the convergence of the matrix elements.

---

(19) We mean precisely that there exist elements of $\tilde{B}_D$ whose corresponding sesquilinear form F (in the sense of Theorem 6) is such that there exists no linear operator A from D into $H = \hat{D}$ for which

$$\forall\, \varphi, \psi \in D\,, \qquad F(\varphi, \psi) = (A\varphi \,|\, \psi)\, .$$

Let us then consider the sequence $(F^{(n)})$ of the sesquilinear forms that are represented by the matrices

$$(F^{(1)}_{\mu\nu}) = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdot \\ 0 & 0 & 0 & 0 & \cdot \\ 0 & 0 & 0 & 0 & \cdot \\ 0 & 0 & 0 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}, \qquad F(^{(2)}_{\mu\nu}) = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdot \\ 1 & 0 & 0 & 0 & \cdot \\ 0 & 0 & 0 & 0 & \cdot \\ 0 & 0 & 0 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix},$$

$$(F^{(3)}_{\mu\nu}) = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdot \\ 1 & 0 & 0 & 0 & \cdot \\ 1 & 0 & 0 & 0 & \cdot \\ 0 & 0 & 0 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}, \ldots .$$

It is immediately seen that such forms correspond, with reference to Theorem 6, to bounded operators $(A^{(n)})$ of D, with an adjoint in D, hence to elements of B_D.

Moreover the matrix elements of the above sequence of matrices converge, so that the sequence $(F^{(n)})$ converges in the (complete HAUSDORFF) space S_D to the sesquilinear form F that is represented by the matrix

$$(F_{\mu\nu}) = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdot \\ 1 & 0 & 0 & 0 & \cdot \\ 1 & 0 & 0 & 0 & \cdot \\ 1 & 0 & 0 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}.$$

So, according to Theorem 6, $(A^{(n)})$ is a CAUCHY sequence in B_D and the form F corresponds to its limit within $\tilde{B}_D$. It is easily seen that the form F represented by the above matrix cannot correspond to any operator with values in H: in fact it does not satisfy the necessary condition that the columns of the representing matrix represent vectors of H.

So it is proved that the element F of S_D corresponds, according to Theorem 6, to an element of $\tilde{B}_D$ which does not represent any operator with values in H.

References.

[1] R. Ascoli, G. Epifanio, A. Restivo, *On the mathematical description of quantized fields*, *Commun.* Math. Phys. **18** (1970), 291-300.

[2] G. Lassner, *Topological Algebras of Operators* (pre-print).

[3] F. Treves, *Topological Vector Spaces, Distributions and Kernels*, Academic Press, New York 1967.

[4] N. Bourbaki, *Elements de Mathématique*: *Espaces Vectoriels Topologiques*, Ch. I et II, 2° ed., Hermann, Paris 1966.

[5] M. A. Naĭmark, *Normed Rings*, Nordhoff, Groningen 1959.

[6] F. Riesz, B. S. Nagy, *Leçons d'Analyse Fonctionnelle*, Gauthier Villars, Paris; Akadémiai Kiado, 4° ed., Budapest 1965.

[7] R. T. Powers, *Self-adjoint algebras of unbounded operators*, Commun. Math. Phys. **21** (1971), 85-124.

[8] T. Kato, *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin, Heidelberg; New York 1966.

[9] N. Bourbaki, *Elements de Mathématique*: *Espaces Vectoriels Topologiques*, Cap. III, IV, V, Hermann, Paris 1966.

[10] R. F. Streater, A. S. Wightman, *PCT, Spin and Statistics, and all that*, Benjamin, New York and Amsterdam 1964.

Summary.

We consider the particularly simple topological $*$-algebra of unbounded operators of any (not necessarily complete) scalar product space D, that consists of the space of the linear operators that are defined on D and have an adjoint in D, endowed with the weak topology. We had already introduced this algebra in a previous paper: here we study some easy properties of it, of its completion and of its sequential completion.

Sommario.

Per ogni spazio D dotato di prodotto scalare (non necessariamente completo) si prende in esame una $*$-algebra topologica di operatori non limitati particolarmente semplice, già introdotta in un precedente lavoro: essa è costituita dallo spazio degli operatori lineari definiti su D ed aventi un aggiunto in D, dotato della topologia debole. Nel presente lavoro si studiano alcune facili proprietà di tale algebra, del suo completamento e del suo completamento sequenziale.

* * *