Riv. Mat. Univ. Parma (3) 2 (1973), 233-239

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Some Results on BV-ω Points. (**)

Let a real valued function $\omega(x)$ be non-decreasing on the closed interval $[a, b]$. Outside the interval, $\omega(x)$ is defined by

$$\omega(x) = \omega(a) \qquad \text{for } x < a \, ,$$

$$\omega(x) = \omega(b) \qquad \text{for } x > b \, .$$

Let S and D denote respectively the set of points of continuity and of discontinuity of $\omega(x)$. Prof. R. L. JEFFERY [4] has defined the class $\mathscr{U}$ of functions $f(x)$ as follows:

$f(x)$ is defined on the set $S[a, b]$ and $f(x)$ is continuous at each point of $S[a, b]$ with respect to the set S. If $x_0 \in D$, then $f(x)$ tends to limits as x tends to x_0+ and x_0- over the points of the set S. These limits will be denoted by $f(x_0+)$ and $f(x_0-)$ respectively. When $x < a$, $f(x) = f(a+)$, and $f(x) = f(b-)$ when $x > b$. $f(x)$ may or may not be defined at the points of D.

Suppose $\mathscr{U}_0 \subset \mathscr{U}$ contains those functions $f(x)$ in $\mathscr{U}$ such that for $x_0 \in D$, both $f(x_0+)$ and $f(x_0-)$ are finite.

Any set of points

$$a \leqslant x_0 < x_1 < x_2 \ldots < x_n \leqslant b$$

such that

$$\omega(x_{i-1}) \neq \omega(x_i) \qquad (i = 1, 2, \ldots, n) \, ,$$

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is called an ω-subdividion [1] of $[a, b]$. In [1] the following definition has been introduced:

A function $f(x)$ in class $\mathscr{U}_0$ is said to be of bounded variation relative to ω, BV-ω, on $[a, b]$ if the least upper bound, $V_\omega(f; [a, b])$, of the sums

$$\sum_{i=1}^{n} |f(x_i+) - f(x_{i-1}-)|$$

is finite for all possible ω-subdivisions $x_0, x_1, \dots, x_n$ of $[a, b]$. The least upper bound, $V_\omega(f; [a, b])$, is called the total ω-variation of $f(x)$ on $[a, b]$.

If $\omega(x)$ is constant in $[a, b]$ then any function in $\mathscr{U}_0$ is assumed to be BV-ω on $[a, b]$. In this case we take $V_\omega(f; [a, b])$ to be equal to zero.

For a function $f(x)$ in $\mathscr{U}_0$ we now introduce the following definition:

Definition. Let x be a point in $[a, b]$. If there exists a closed neighbourhood of x in which $f(x)$ is of bounded variation relative to ω, then we say that x is a point of bounded variation relative to ω of $f(x)$. On the otherhand, if there is no closed neighbourhood of x in which $f(x)$ is BV-ω, then x is said to be a point of non-bounded variation relative to ω of $f(x)$.

The points of the former type will, in short, be denoted by BV-ω points, and of the latter type by NBV-ω points. It easily follows that the set of BV-ω points of $f(x)$ is either void or an open set, and so the set of NBV-ω points is closed relative to $[a, b]$.

Now, let x be an interior point of $[a, b]$. Choose two positive numbers δ_1 and δ_2 such that

$$[x - \delta_1, x + \delta_2] \subset [a, b] .$$

Let

$$\theta\,(x, \delta_1, \delta_2) = \frac{1}{1 + V_\omega(f;\ [\alpha - \delta_1, \alpha + \delta_2])}\,, \tag{1}$$

where $V_\omega(f; [x - \delta_1, x + \delta_2])$ denotes the total ω-variation of $f(x)$ in $[x - \delta_1, x + \delta_2]$. The function $\theta(x, \delta_1, \delta_2)$ is monotone nondecreasing as δ_1 and δ_2 tend to zero. We denote the limit

$$\lim_{\substack{\delta_1 \to 0 \\ \delta_2 \to 0}} \theta(x, \delta_1, \delta_2)\,,$$

by $\theta(x)$. If x is an endpoint of $[a, b]$, the function $\theta(x)$ is defined by taking a closed neighbourhood of x in its usual restricted way. The function $\theta(x)$ has then a definite value at each point x of $[a, b]$ and $0 \leqslant \theta(x) \leqslant 1$.

It follows easily that if $\theta(x) = 0$ at some point $x \in [a, b]$, then x is a NBV-ω point of $f(x)$, while if for a point x

$$\theta(x) \neq 0$$

then x is a BV-ω point of $f(x)$.

The purpose of the present paper is to study certain properties of the set of BV-ω points and also to establish certain properties of $\theta(x)$. Throughout the paper the following notations will be used:

S_0 denotes the union of pairwise disjoint open intervals (a_i, b_i) in $[a, b]$, on each of which $\omega(x)$ is constant;

$$S_1 = \{a_1, b_1, a_2, b_2, \ldots\};$$

$$S_2 = SS_1;$$

and

$$S_3 = S[a, b] - (S_0 + S_2).$$

Theorem 1. *The function $\theta(x)$ is lower semi-continuous in $[a, b]$.*

Proof. Let α be a point of the open interval (a, b) and $\varepsilon > 0$ be arbitrary. There exist two positive numbers δ_1 and δ_2 such that

$$\theta(\alpha, \delta_1, \delta_2) > \theta(\alpha) - \varepsilon \,. \tag{2}$$

Choose two positive numbers δ_2' and δ_1' such that

$$[x - \delta_1', x + \delta_2'] \subset (\alpha - \delta_1, \alpha + \delta_2) \,.$$

Since

$$V_\omega(f; [x - \delta_1', x + \delta_2']) \leqslant V_\omega(f; [\alpha - \delta_1, \alpha + \delta_2]) \,,$$

it follows from relation of the form (1) that,

$$\theta(x) \geqslant \theta(x, \delta_1', \delta_2') \geqslant \theta(\alpha, \delta_1, \delta_2) \,. \tag{3}$$

Combining (2) and (3) we see that for every point $x \in (\alpha - \delta_1, \alpha + \delta_2)$,

$$\theta(x) > \theta(\alpha) - \varepsilon .$$

Hence $\theta(x)$ is lower semi-continuous at α, and so in (a, b). Similarly we can show one sided lower semi-continuity of $\theta(x)$ at the end-points a and b.

This proves the theorem.

Theorem 2. *If $\theta(x) = 1$ for every $x \in D$, then $f(x)$ is continuous on $[a, b]$.*

Proof. Let $x_0 \in D$. Then

$$\theta(x_0) = 1$$

and so there exists a $\delta_0 > 0$ such that $f(x)$ is BV-ω in $[x_0 - \delta_0, x_0 + \delta_0]$.

Again $\theta(x_0) = 1$ implies that

$$\lim_{\substack{\delta_1 \to 0 \\ \delta_2 \to 0}} V_\omega(f; [x_0 - \delta_1, x_0 + \delta_2]) = 0$$

and so, for arbitrary $\varepsilon > 0$ there exist two positive numbers δ_1' and δ_2' with $0 < \delta_1' < \delta_0$, $0 < \delta_2' < \delta_0$ such that

$$V_\omega(f; [x_0 - \delta_1', x_0 + \delta_2']) < \varepsilon .$$

Hence for any two points ξ, η of S with

$$x_0 - \delta_1' \leqslant \xi < x_0 < \eta \leqslant x_0 + \delta_2', \qquad |f(\eta) - f(\xi)| < \varepsilon .$$

Letting $\xi \to x_0-$ and $\eta \to x_0+$ over the points of S we get

$$|f(x_0+) - f(x_0-)| \leqslant \varepsilon .$$

Since $\varepsilon > 0$ is arbitrary,

$$f(x_0+) = f(x_0-) .$$

If at points of D we set

$$f(x) = f(x+) = f(x-) ,$$

then $f(x)$ is continuous on $[a, b]$. This proves the theorem.

Theorem 3. *If $\alpha \in S_3$ is a limiting point of S_3 on both sides and if α is a BV-ω point of $f(x)$, then $\theta(\alpha) = 1$.*

Proof. Since α is a BV-ω point of $f(x)$, there exists a $\delta > 0$ such that $f(x)$ is BV-ω in $[\alpha - \delta, \alpha + \delta]$.

Let

$$\pi(t) = V_\omega(f; [\alpha - \delta, t]), \qquad t \in [\alpha - \delta, \alpha + \delta].$$

Then $\pi(t)$ is continuous on S_3 ([3], Lemma 2.2).

Choose two positive numbers δ_1 and δ_2 such that the points $\alpha - \delta_1$ and $\alpha + \delta_2$ belong to $S_3[\alpha - \delta, \alpha + \delta]$.

Then

$$V_\omega(f; [\alpha - \delta_1, \alpha + \delta_2]) = \pi(\alpha + \delta_2) - \pi(\alpha - \delta_1) \qquad ([3], \text{ Lemma 2.1}).$$

So,

$$\theta(\alpha, \delta_1, \delta_2) = \frac{1}{1 + V_\omega f; [x - \delta_1, x + \delta_2])} = \frac{1}{1 + \pi(\alpha + \delta_2) - \pi(\alpha - \delta_1)}.$$

Letting $\delta_1 \to 0$ and $\delta_2 \to 0$ with $\alpha - \delta_1 \in S_3$, $\alpha + \delta_2 \in S_3$, we get from the above,

$$\theta(\alpha) = 1.$$

This completes the proof.

Theorem 4. *If E denotes the set of BV-ω points in $S[a, b]$, then*

$$|E|_\omega = (LS) \int_{S[a,b]} \theta(x)\, d\omega,$$

where $|E|_\omega$ denotes ω-mesure [4] of E and the integral is taken in Lebesgue-Stieltjes sense ([4], [2]).

Proof. Since $\theta(x)$ is lower semi-continuous in $[a, b]$, it is ω-mesurable [4] in $[a, b]$. Moreover, $\theta(x)$ is bounded on $S[a, b]$ and so, $\theta(x)$ is LEBESGUE-STIELTJES integrable in $S[a, b]$.

Again E is the set of points x in $S[a, b]$ for which

$$\theta(x) > 0,$$

and so, the set E is ω-measurable.

So,

$$(4) \qquad (LS)\int_{S[a,b]} \theta(x)\,d\omega = (LS)\int_{E} \theta(x)\,d\omega + (LS)\int_{S[a,b]-E} \theta(x)d\omega\,.$$

Let F denote the set of points of ES_3 which are limit points of ES_3 on both sides. Since $ES_3 - F$ is at most enumerable,

$$|ES_3|_\omega = |F|_\omega\,.$$

Now,

$$|S_0|_\omega = 0\,, \qquad |S_2|_\omega = 0\,.$$

Also,

$$\theta(x) = 0 \qquad \text{for } x \in S[a, b] - E\,,$$

$$\theta(x) = 1 \qquad \text{for } x \in F\,.$$

So, from (4) we get

$$(LS)\int_{S[a,b]} \theta(x)\,d\omega = (LS)\int_{ES_3} \theta(x)\,d\omega + (LS)\int_{ES_0} \theta(x)\,d\omega + (LS)\int_{ES_2} \theta(x)d\omega$$
$$= (LS)\int_{ES_3} \theta(x)\,d\omega = (LS)\int_{F} \theta(x)\,d\omega = |F|_\omega = |ES_3|_\omega = |E|_\omega\,.$$

Theorem 5. *If $\omega(x)$ is strictly increasing and continuous in $(\alpha, \beta) \subset [a, b]$, then the set of points in (α, β) of discontinuity of $\theta(x)$ is non-dense in $[\alpha, \beta]$.*

Proof. Let A denote the set of points of discontinuity of $\theta(x)$ in (α, β) and let $x_0 \in (\alpha, \beta)$ be a BV-ω point of $f(x)$. By Theorem 3,

$$\theta(x_0) = 1\,.$$

Since $\theta(x)$ is lower semi-continuous and $0 \leqslant \theta(x) \leqslant 1$, it follows that $\theta(x)$ is continuous at x_0. Hence each point of A is a NBV-ω point of $f(x)$.

If possible, suppose A is not non-dense in $[\alpha, \beta]$. Then A is everywhere dense in an interval $[\alpha', \beta'] \subset [\alpha, \beta]$. Denote $A[\alpha', \beta']$ by A_1. Since the set of NBV-ω points of $f(x)$ is closed, it follows that A_1 is a closed set. Again since A_1 is everywhere dense in $[\alpha', \beta']$, every point of $[\alpha', \beta']$ is a NBV-ω point of $f(x)$.

So,

$$\theta(x) = 0 \qquad \text{for } x \in [\alpha', \beta'] .$$

Hence $\theta(x)$ is continuous in (α', β') which contradicts the definition of A and completes the proof of the theorem.

I am grateful to Dr. M. C. CHAKRABARTY for his kind help and suggestions in the preparation of the paper.

References.

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