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On the Product of Two Appel Polynomials. (**)

Introduction.

L. CARLITZ [1] found some results concerning the product of two APPELL polynomials. In this Note we propose to find the condition that the product of two polynomials be an APPELL polynomial.

An APPELL polynomial has the important property of reproducing itself under the operation of differentiation, in accordance with the rule

(1) $$\frac{\mathrm{d}}{\mathrm{d}x} P_n(x) = n P_{n-1}(x) ,$$

where $P_n(x)$ is a polynomial in x of degree n. An equivalent definition is the existence of a formal power series

(2) $$A(t) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!} , \quad a_0 \neq 0 ,$$

such that

(3) $$A(t) \exp(xt) = \sum_{n=0}^{\infty} P_n(x) t^n ,$$

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so that

$$P_n(x) = \sum_{r=0}^{n} \binom{n}{r} a_{n-r} x^r . \tag{4}$$

In the first part of this note, we consider the product of two arbitrary APPELL polynomials $F_n(x)$ and $G_m(x)$ where $F_n(x)$ is independent of m and $G_m(x)$ is independent of n.

In the second section we consider the product of two arbitrary polynomials (not necessarily of APPELL type).

1. – Let $F_n(x)$ be an APPELL polynomial of degree n, defined by

$$F_n(x) = \sum_{r=0}^{n} \binom{n}{r} f_{n-r} x^r$$

and let $G_m(x)$ be another APPELL polynomial of degree m, defined by

$$G_m(x) = \sum_{s=0}^{n} \binom{m}{s} g_{m-s} x^s .$$

Let us consider the product

$$S_{m+n}(x) = F_n(x)\, G_m(x) = \sum_{s=0}^{m+n} x^s \sum_{r=0}^{n} \binom{n}{r} \binom{m}{s-r} f_{n-r}\, g_{m-s+r} ,$$

or

$$S_p(x) = \sum_{s=0}^{p} x^s \sum_{r=0}^{n} \binom{n}{r} \binom{p-n}{s-r} f_{n-r} g_{p-n-s+r} .$$

If the product $S_{m+n}(x)$ of the two polynomials $F_n(x)$ and $G_m(x)$ be also an APPELL polynomial, then $S_p(x)$ must have a generating function of the type

$$A(t) \exp(xt) = \sum_{p=0}^{\infty} S_p(x) \frac{t^p}{p!} , \tag{5}$$

where $A(t)$ is a power series in t. Now

$$(6)\quad \left\{\begin{aligned} \sum_{p=0}^{\infty} S_p(x)\frac{t^p}{p!} &= \sum_{p=0}^{\infty}\sum_{s=0}^{p} x^s \sum_{r=0}^{n}\binom{n}{r}\binom{p-n}{s-r} f_{n-r}\, g_{p-n-s+r}\frac{t^p}{p!} = \\ &= \sum_{s=0}^{\infty}\frac{(xt)^s}{s!}\sum_{p=0}^{\infty}\sum_{r=0}^{n} s!\binom{n}{y}\binom{p+s-n}{s-r} f_{n-r}\, g_{p-n+r}\frac{t^p.}{(p+s)!}\,. \end{aligned}\right.$$

From (6) we see that the polynomial $S_p(x)$ will have a generating function of the form (5), if and only if the expression

$$\sum_{p=0}^{\infty}\sum_{r=0}^{n}\binom{n}{r}\binom{p+s-n}{s-r}\frac{s!}{(p+s)!} f_{n-r}\, g_{p-n+r}$$

be independent of s, i.e., when $p=0$ and $r=n$ and in this case, (6) gives

$$\sum_{p=0}^{\infty} S_p\ (x)\frac{t^p}{p!} = f_0 g_0 \exp\,(xt)\,.$$

Hence, we conclude

Theorem 1. *If $F_n(x)$ and $G_m(x)$ are two Appell polynomials of degree n and m respectively, where $F_n(x)$ is independent of m and $G_m(x)$ is independent of n, then $F_n(x)\cdot G_m(x)$ will be an Appell polynomial, if and only if, $F_n(x) = x^n$ and $G_m(x) = x^m$.*

2. – The above treatment suggests a natural extension. In this section we seek the condition that the product of any two polynomials (not necessarily of Appell type) may be a polynomial of Appell type.

Let

$$(7)\qquad f_n(x) = \sum_{r=0}^{n} a_{r,m,n}\, x^r$$

and

$$(8)\qquad g_m(x) = \sum_{s=0}^{m} b_{s,m,n}\, x^s$$

be any two polynomials of degree n and m respectively, the coefficients $a_{r,m,n}$

and $b_{s,m,n}$ being functions of both n and m. There is no loss of generality in assuming $n > m$.

Consider the product

$$P_{m+n}(x) = f_n(x)\, g_m(x) = \sum_{r=0}^{n} \sum_{s=0}^{m} a_{r,m,n} b_{s,m,n} x^{r+s}.$$

We have

$$(9)\quad \begin{cases} P_{m+n}(x) = \sum_{r=0}^{n} x^r \sum_{t=0}^{r} a_{t,m,n} b_{r-t,m,n} + \sum_{r=m+1}^{n} x^r \sum_{t=0}^{m} a_{r-t,m,n} b_{t,m,n} \\ \qquad + \sum_{r=n+1}^{m+1} x^r \sum_{t=0}^{m+n-r} a_{n-t,m,n} b_{r-n+t,m,n}. \end{cases}$$

Differentiating (9), we get

$$(10)\quad \begin{cases} P'_{m+n}(x) = \sum_{r=0}^{m-1} (r+1)\, x^r \sum_{t=0}^{r+1} a_{t,m,n} b_{r+1-t,m,n} + \\ + \sum_{r=m}^{n-1} (r+1)\, x^r \sum_{t=0}^{m} a_{r+1-t,m,n} b_{t,m,n} + \sum_{r=n}^{n+m-1} (r+1)\, x^r \sum_{t=0}^{m+n-r-1} a_{n-t,m,n} b_{r+1-n+t,m,n}. \end{cases}$$

Again, from (9), we have

$$(11)\quad \begin{cases} P_{m+n-1}(x) = \sum_{r=0}^{m-1} x^r \sum_{t=0}^{r} a_{t,m-1,n} b_{r-t,m-1,n} + \sum_{r=m}^{n-1} x^r \sum_{t=0}^{m-1} b_{t,m-1,n} a_{r-t,m-1,n} \\ \qquad + \sum_{r=n}^{n+m-1} x^r \sum_{t=0}^{m+n-r-1} a_{n-t,m-1,n} b_{r-n+t,m-1,n}. \end{cases}$$

If $P_{m+n}(x)$ is also an APPELL polynomial, we must have

$$(12)\qquad \frac{d}{dx} P_{m+n}(x) = (m+n)\, P_{m+n-1}(x).$$

Hence, if $P_{m+n}(x)$ is an APPELL polynomial, we must have

(13a) $$(r+1)\sum_{t=0}^{r+1} a_{t,m,n}b_{r+1-t,m,n} = (m+n)\sum_{t=0}^{r} a_{t,m-1,n}b_{r-t,m-1,n} \quad (0 \leqslant r \leqslant m-1),$$

(13b) $$(r+1)\sum_{t=0}^{m} a_{r+1-t,m,n}b_{t,m,n} = (m+n)\sum_{t=0}^{m-1} a_{r-t,m-1,n}b_{t,m-1,n} \quad (m \leqslant r \leqslant n-1),$$

(13c) $$(r+1)\sum_{t=0}^{m+n-r-1} a_{n-t,m,n}b_{r+1-n+t,m,n} = (m+n)\sum_{t=0}^{m+n-r-1} a_{n-t,m-1,n}b_{r-n+t,m-1,n} \quad (n \leqslant r \leqslant m+n-1).$$

Thus, a necessary condition that the product of the two polynomials $f_n(x)$ and $g_m(x)$ of degree n and m respectively, be an APPELL polynomial of degree $n+m$ is that the relations (13a) and (13c) must hold.

It is interesting to note that (13a) and (13c) may be written as

(14*a*) $$\sum_{t=0}^{r+1} a_{t,m,n}b_{r+1-t,m,n} = \frac{(m+n)!}{(r+1)!\,(m+n-r-1)!}\, a_{0,m-r-1,n}\, b_{0,m-r-1,n} \quad (0 \leqslant r \leqslant m-1),$$

(14*b*) $$\sum_{t=0}^{m} a_{m+r-t,m,n}b_{t,m,n} = \frac{(m+n)!\,r!}{n!\,(m+r)!}\, a_{r,0,n}b_{0,0,n} \quad (1 \leqslant r \leqslant n-m),$$

(14*c*) $$\sum_{t=0}^{m-r} a_{n-t,m,n}b_{r+t,m,n} = \frac{(m+n)!\,(n-m+r)!}{n!\,(n+r)!}\, a_{n-m+r,0,n}b_{0,0,n} \quad (1 \leqslant r \leqslant m).$$

Special cases.

Let

$$g_m(x) = x^m = \sum_{s=0}^{m} b_{s,m,n}x^s .$$

We see that $b_{m,m,n} = 1$; $b_{s,m,n} = 0$, $s \neq m$ (for all values of m). Hence, we must have in this case

(i) $$b_{m,m,n} = b_{m-1,m-1,n} = b_{m-2,m-2,n} = \dots = b_{0,0,n} = 1 .$$

In this case (14a) and (14c) reduce to

(ii) $$a_{r,m,n} = \frac{(m+n)!\,r!}{n!\,(m+r)!}\, a_{r,0,n} \qquad (0 \leqslant r \leqslant n)\,.$$

Hence, the polynomial $f_n(x)$ must be given by

(iii) $$f_n(x) = \sum_{r=0}^{n} a_{r,m,n}\, x^r = \sum_{r=0}^{n} \frac{(m+n)!\,r!}{n!\,(m+r)!}\, a_{r,0,n}\, x^r\,.$$

In general

(iv) $$f_n(x) = (m+n)! \sum_{r=0}^{n} \frac{a_r\, x^r}{(m+r)!}$$

where a_r is independent of m, will be an APPELL polynomial when multiplied by x^m. In particular, if

$$a_r = \frac{(-1)^r\, n!}{[r!]^2\,(n-r)!}$$

then, it is easy to see that $x^m L_n^{(m)}(x)$ will be an APPELL polynomial, where $L_n^{(m)}(x)$ is the LAGUERRE polynomial.

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Reference.

[1] L. CARLITZ, *Product of two Appell polynomials*, Colloq. Math. (3) **15** (1963), 245-258.

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