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An Extension of a Class of Polynomials - III. (**)

Introduction.

About half a century ago NIELSEN [6] studied a remarkable subset of the APPELL set by considering a set of polynomials $\{p_n(x)\}$ which satisfies the two functional equations, for $n = 0, 1, 2, \ldots$

$$\frac{\mathrm{d}}{\mathrm{d}x} P_n(x) = P_{n-1}(x)\,, \qquad P_n(-n-1) = (-1)^n P_n(x) \tag{1}$$

and showed their importance in the theory of BERNOULLI and EULER numbers. A few years later WARD [10] generalized the set by considering the polynomials $\{Y_n(x)\}$ for which

$$\frac{\mathrm{d}}{\mathrm{d}x} Y_n(x) = Y_{n-1}(x)\,, \qquad Y_n(ax+b) = \tau_n Y_n(x) \quad \text{for } n = 0, 1, 2, \ldots \tag{2}$$

where a, b are any two complex numbers. More recently, SHARMA and CHAK [8] replaced the differential operator by the q-difference operator of JACKSON and obtained properties analogous to the properties of the subsets studied by NIELSEN and by WARD (see also [1]). Incidentally, the q-difference operator of JACKSON is given by

$$D_q \equiv \frac{q^{x\frac{d}{dx}} - 1}{(q-1)x} \qquad \text{or} \qquad D_q f(x) = \frac{f(qx) - f(x)}{(q-1)x} \tag{3}$$

and it tends to the ordinary differential operator $D \equiv \mathrm{d}/\mathrm{d}x$ as $q \to 1$.

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In the previous two papers of this series of papers CHAK [2] and CHAK and AGARWAL [3] studied the APPELL set of polynomials to the base (u), that is, the class of polynomials $\{H_n(x)\}$ in x which satisfy the functional equation

(4) $$D_u H_n(x) = H_{n-1}(x)\,, \qquad n = 0, 1, 2, \ldots$$

where D_u is a general operator, linear and distributive, which transforms a polynomial of degree n in x into one of degree $n-1$. In particular, $D_u x^n = u_n x^{n-1}$ where (u) is a given sequence of real or complex numbers subject to the only restrictions that $u_0 = 0$, $u_1 = 1$, and $u_n \neq 0$ for $n \geqslant 1$; (see WARD [11]). We know that (see [2] with Δu_n instead of u^n) $D_u \equiv D_q$ if and only if the sequence $\{\Delta u_n\}$ obeys the law $\Delta u_n \cdot \Delta u_m = \Delta u_{n+m}$ for $n, m = 0, 1, 2, \ldots$, where $\Delta u_n \equiv u_{n+1} - u_n$.

1. – u-regular polynomial sets. In the present paper we wish to study the properties of the system of polynomials $\{H_n(x)\}$ which satisfies either of the following two sets of functional equations $(n = 0, 1, 2, \ldots)$

(1.1) $$D_u H_n(x) = H_{n-1}(x)\,, \qquad H_n[ax + b] = \tau_n H_n(x)$$

(1.2) $$D_u H_n(x) = H_{n-1}(x)\,, \qquad H_n[b + ax] = \tau_n H_n[x]$$

where (see [2], [3])

(i) a and b are any complex or real numbers;

(ii) $u_n! = u_n \cdot u_{n-1} \ldots u_1$ where n is a positive integer, $u_n! = 1$ if $n = 0$;

(iii) the APPELL set of polynomials $\{H_n(x)\}$ to the base (u) is given by

$$H_n(x) = \sum_{i=0}^{n} \frac{h_{n-i}}{u_i!}\, x^i\,, \qquad H_n(0) = h_n\,;$$

(iv) $$H_n[x + b] = \sum_{i=0}^{n} \frac{x^i}{u_i!}\, H_{n,i}[b] = \sum_{i=0}^{n} \frac{x^i}{u_i!} \sum_{r=0}^{n-i} h_r \alpha_{n-r,n-i-r} b^{n-i-r}\,,$$

$H_{n,0}[x] = H_n[x]$;

(v) $$H_n[b + x] = \sum_{i=0}^{n} x^i H_{n,i}(b) = \sum_{i=0}^{n} x^i \sum_{r=0}^{n-i} \frac{h_r}{u_{n-i-r}!}\, \alpha_{n-r,i}\, b^{n-i-r}\,,$$

$H_{n,0}(b) = H_n(b)$;

(vi) $\alpha_{n,0} = 1$ for $n = 0, 1, 2, \dots,$

$$\alpha_{n,n} = \prod_{r=0}^{n-1} \frac{(u_{r+1} - u_r)}{u_n!} \qquad \text{for } n = 1, 2: \dots,$$

$$u_n \alpha_{n,r} = u_{n-r} \alpha_{n-1,r} + (u_n - u_{n-1}) \alpha_{n-1,r-1} \qquad \text{for } r = 0, 1, 2, \dots, n \text{ and } n = 1, 2, \dots.$$

We call the subset satisfying (1.1) « *u* (*I*)*-regular* » *and the subset satisfying* (1.2) « *u* (*II*)*-regular* ».

Obviously these sets are subsets of the polynomial system already studied by CHAK [2] and have properties analogous to the regular and cyclic sets of NIELSEN [6] and of WARD [10]; (see also CHAK [1], SHARMA and CHAK [8]).

Leaving the trivial case where $a = 0$ or $a = 1$ we have the following easily-proved theorems:

Theorem 1. *A necessary and sufficient condition for a sequence of polynomials* $\{H_n(x)\}$ *to be* $u(I)$*-regular is*

$$H_{n,r}[b] = a^{n-r} H_{n-r}(0) \qquad \textit{for } r = 0, 1, 2, \dots, n \tag{1.3}$$

provided that a is not a root of unity.

By solving these equations we have

$$h_n = \frac{h_0 b^n \Delta_n}{(a-1)(a^2-1) \dots (a^n-1)} \qquad (n = 1, 2, \dots) \tag{1.4}$$

where Δ_n is the n-th order determinant whose first row has the elements $\alpha_{n,1}, 1 - a, 0, \dots, 0,$ the second row the elements $\alpha_{n,2}, \alpha_{n-1,1}, 1 - a^2, 0, \dots, 0$ and so on until we come to the last two rows having the elements

$$a_{n,n-1}, \quad \alpha_{n-1,n-2}, \quad \dots, \quad \alpha_{2,1}, \quad 1 - a^{n-1},$$

and

$$a_{n,n}, \quad \alpha_{n-1,n-1'}, \quad \dots, \quad \alpha_{2,2}, \quad \alpha_{1,1}.$$

Theorem 2. *A necessary and sufficient condition for a sequence of polynomials* $\{H_n(x)\}$ *to be* $u(II)$*-regular is*

$$\frac{1}{\alpha_{r,r}} H_{n,r}(b) = a^{n-r} H_{n-r}(0) \qquad \text{for } r = 0, 1, 2, \dots, n \tag{1.5}$$

provided that a is not a root of unity.

The solution in this case is

$$h_n = \frac{h_0 b^n \Delta_n'}{(a-1)(a^2-1)\dots(a^n-1)} \qquad (n = 1, 2, \dots) \tag{1.6}$$

where Δ_n' is the n-th order determinant whose first row has the elements $\frac{\alpha_{n,n-1}}{\alpha_{n-1,n-1}}$, $(1-a)$, $0, \dots, 0$, the second row the elements $\frac{\alpha_{n,n-2}}{\alpha_{n-2,n-2}} \cdot \frac{1}{u_2!}$, $\frac{\alpha_{n-1,n-2}}{\alpha_{n-2,n-2}}$, $(1-a^2)$, $0, \dots, 0$, and so on until we come to the last two rows having elements

$$\frac{\alpha_{n,1}}{\alpha_{1,1}} \cdot \frac{1}{u_{n-1}!}, \; \frac{\alpha_{n-1,1}}{\alpha_{1,1}} \cdot \frac{1}{u_{n-2}!}, \dots, \frac{\alpha_{2,1}}{\alpha_{1,1}}, \; (1-a^{n-1}) \text{ and } \frac{1}{u_n!}, \frac{1}{u_{n-1}!}, \dots, \frac{1}{u_2!}, \frac{1}{u_1!}.$$

We thus see that if a is not a root of unity, the polynomials $\{H_n(x)\}$ are completely determined in terms of h_0 if they satisfy either of the two set of functional equations i.e. *for both $u(I)$-regular and $u(II)$-regular sequence of polynomials $h_1, h_2, \dots$ are finite only if a is not a root of unity. If, howerver, $a^r = 1$ $[r \equiv 0 \pmod p]$, then it is easy to see that an infinite sequence of polynomials $\{H_n(x)\}$ does not exist in either of the two cases.*

We next take up the case when a is a root of unity. Let us introduce two special triangular matrices T and T' of non-zero numbers given by

$$T \equiv \left\{ \begin{array}{lllll} c_{0,0} & & & & \\ c_{1,0} & c_{0,1} & & & \\ c_{2,0} & c_{1,1} & c_{0,2} & & \\ \cdots & \cdots & \cdots & \cdots & \\ c_{n,0} & c_{n-1,1} & c_{n-2,2} & \cdots & c_{0,n} \end{array} \right.$$

and

$$T' \equiv \left\{ \begin{array}{lllll} c_{n,0} & & & & \\ c_{n-1,0} & c_{n-1,1} & & & \\ c_{n-2,0} & c_{n-2,1} & c_{n-2,2} & & \\ \cdots & \cdots & \cdots & \cdots & \\ c_{0,0} & c_{0,1} & c_{0,2} & \cdots & c_{0,n} \end{array} \right.$$

where $c_{n,0} = 1$ for $n = 0, 1, 2, \dots$.

We now use the following notations;

$$H_n(T;\,x) = \sum_{i=0}^{n} \frac{h_i c_{n-i,i}}{u_{n-i}!}\, x^{n-i}\;;$$

$$H_n(T;\;ax+b) = \sum_{i=0}^{n} \frac{h_i}{u_{n-i}!}\,(ax+b)_{T_{n-i}}\,,$$

where

$$(ax+b)_{T_n} = \sum_{i=0}^{n} (ax)^{n-i} b^i c_{n-i,i}\,;$$

$$H_n(T'_k;\,b) = \sum_{i=0}^{n} \frac{h_i}{[u_{n-i}]_{k-i}}\, b^{k-i} c_{n-k,\,k-i}$$

and

$$[u_n]_s = u_n u_{n-1} \dots u_{n-s+1}\,, \qquad [u_n]_0 = 1\,.$$

It is easy to see that

$$H_n(T'_n;\,b) = \sum_{i=0}^{n} \frac{h_i}{u_{n-i}!}\, b^{n-i} c_{0,n-i}\;; \tag{1.7}$$

$$H_n(T'_0;\,b) = h_0 c_{n,0} = h_0\;; \tag{1.8}$$

$$H_n(T;\,ax+b) = \sum_{i=0}^{n} \frac{(ax)^{n-i}}{u_{n-i}!}\, H_n(T'_i;\,b)\,. \tag{1.9}$$

Now let us consider the class of polynomials $\{H_n(x)\}$ which satisfies the two functional equations $(n = 0, 1, 2, \dots)$

$$D_u H_n(x) = H_{n-1}(x) \quad \text{and} \quad H_n(T;\,ax+b) = \tau_n H_n(x)\,. \tag{1.10}$$

We at once have for $i = 0, 1, 2, \dots, n$

$$\tau_n = a^n \quad \text{and} \quad H_n(T'_i;\,b) = a^i H_i(0)\,. \tag{1.11}$$

These equations easily give $\{h_n\}$ (when a is not a root of unity) in the form

$$h_n = \frac{h_0 b^n \Delta_n(a,\,T)}{(a-1)(a^2-1)\dots(a^n-1)} \qquad (n = 1, 2, \dots)\,, \tag{1.12}$$

where $\Delta_n(a, T)$ is the n-th order determinant whose first row has the elements $\frac{c_{n-1,1}}{[u_n]_1}$, $(1-a)$, 0, ..., 0, the second row the elements $\frac{c_{n-2,2}}{[u_n]_2}$, $\frac{c_{n-2,1}}{[u_{n-1}]_1}$, $(1-a^2)$, 0, ..., 0 and so on until we come to the two rows having elements $\frac{c_{1,n-1}}{[u_n]_{n-1}}$, $\frac{c_{1,n-2}}{[u_{n-1}]_{n-2}}$, ..., $(1-a^{n-1})$ and $\frac{c_{0,n}}{[u_n]_n}$, $\frac{c_{0,n-1}}{[u_{n-1}]_{n-1}}$, ..., $c_{0,1}$.

Now suppose we have $a \neq 1$ but $a^3 = 1$, that is $a = \exp(2\pi i/3)$ or $a = \exp(2 \cdot 2\pi i/3)$. For these values of a the 3 equations given by (1.11) should be consistent. Also if $a^3 = 1$, then $a^6 = 1$, $a^9 = 1$, ... This suggests

Theorem 3. *When $a^r = 1$ $[r \equiv 0 \pmod p]$, for $p < n$, a necessary and sufficient condition that equations* (1.11) *be consistent is that the following determinant of order p vanishes identically for $a = \exp(2\pi ik/p)$ $(k = 1, 2, ..., p-1)$:*

$$(1.13)\qquad \begin{vmatrix} \frac{c_{n-1,1}}{[u_n]_1} & 1-a & 0 & \cdot & \cdot & 0 \\ \frac{c_{n-2,2}}{[u_n]_2} & \frac{c_{n-2,1}}{[u_{n-1}]_1} & 1-a^2 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \frac{c_{n-p,p}}{[u_n]_p} & \frac{c_{n-p,p-1}}{[u_{n-1}]_{p-1}} & \cdot & \cdot & \cdot & \frac{c_{n-p,1}}{[u_{n-p+1}]_1} \end{vmatrix}$$

If all the determinants of order p, $2p$, $3p$, ..., vanish identically when $a^r = 1$ $[r \equiv 0 \pmod p]$ then there exist solutions giving h_n for all n.

Furthermore, if the elements of the matrix T satisfy the condition

$$(1.14)\qquad \frac{c_{n-k,k}}{[u_n]_k} = \frac{c_{0,k}}{u_k!} \qquad \text{for } k = 1, 2, \dots, n$$

then we say that the set of polynomials $\{H_n(x)\}$ are « $u(T)$-regular » and the following theorem is easy to prove:

Theorem 4. *If a is not a root of unity then there is a unique $u(T)$-regular sequence and a necessary and sufficient condition for a sequence to be $u(T)$-regular is*

$$(1.15)\qquad H_n(T_n'; b) = a^n H_n(0) \qquad \textit{for } n = 0, 1, 2, \dots .$$

If however, a is a primitive p-th root of unity ($\neq 1$) *then there are infinitely many* $u(T)$*-regular sequences, called hereafter « $u(T)$-cyclic sequences of order p », only if the matrix T is such that the determinants of the type* (1.13) *of order p, $2p$, $3p$, ... all vanish identically when* $a^r = 1$ $[r \equiv 0 \pmod{p}]$.

A particular case of the $u(T)$-regular sequence is the sequence obtainable from the former if one takes

$$c_{0,k} = \prod_{r=0}^{k-1} (u_{r+1} - u_r) \qquad \text{for } k = 1, 2, \ldots, n$$

and in this case a necessary and sufficient condition is

$$H_n[b] = a^n H_n(0) \qquad \text{for } n = 1, 2, \ldots . \tag{1.16}$$

For another example of a $u(T)$-regular sequence take $c_{0,k} = \dfrac{u_k!}{k!}$ for $k = 1, 2, \ldots, n$. In this case

$$h_r = \frac{h_0}{r!} \cdot \frac{b^r}{(a-1)^r}, \qquad r = 0, 1, 2, \ldots \tag{1.17}$$

for all values of a ($\neq 1$); and if $\lambda = \dfrac{b}{(a-1)}$, then $H_n(x)$ is given by

$$H_n(x) = h_0 \left(\frac{x_n}{u^n!} + \frac{x^{n-1}\lambda}{u_{n-1}!} + \frac{x^{n-2}\lambda^2}{2!\,u_{n-2}!} + \ldots + \frac{\lambda^n}{n!} \right); \tag{1.18}$$

this is the simplest $u(T)$-regular sequence.

For $u(T)$-cyclic sequences it is not at all difficult to prove:

Theorem 5. *If* $K_n(x) = \sum\limits_{i=0}^{n} \dfrac{k_i}{u_{n-i}!} x^{n-i}$ *is a* $u(T)$*-cyclic sequence of order* p *then* $K_n(x)$ *is uniquely represented by*

$$K_n(x) = \sum_{s=0}^{r} \gamma_{sp} H_{n-sp}(x), \tag{1.19}$$

where $\{H_n(x)\}$ *is the simplest* $u(T)$*-regular sequence and* $\gamma_0, \gamma_p, \ldots, \gamma_{rp}$ *are constants* $[rp \leqslant n < (r+1)p]$. *Conversely, every sequence* $K_n(x)$ *of the form* (1.19) *determines a* $u(T)$*-cyclic sequence of order* p.

Since the proof of this theorem is exactly on the lines of Sharma and Chak [**8**] for $q(T)$-cyclic sequences it is omitted.

2. – The $q(I)$-regular, $q(II)$-regular and $q(T)$-cyclic sequences treated by SHARMA and CHAK [**8**] are particular cases of our regular and cyclic sequences if we allow $\Delta u_n (\equiv u_{n+1} - u_n)$ to satisfy the law $\Delta u_n \cdot \Delta u_m = \Delta u_{n+m}$. Further, if $u_n = n$ we obtain the results of WARD [**10**] and WARD's results give the properties of the sequences studied by NIELSEN [**6**] by taking $a = -1 = b$.

It is interesting to note that the orthogonality of polynomials with BRENKE type generating functions, of which the APPELL set of polynomials to the base (u) is a special case, has recently been studied by CHIHARA [**4**]; also that ISMAIL [**5**] has more recently generalized the work of SHEFFER [**9**] on polynomial sets of type zero by replacing the ordinary differential operator D by WARD's operator D_u. ISMAIL [**5**] and SCARAVELLI [**7**] both have given good bibliographies on APPELL polynomials. A lot of the results on APPELL polynomials can be generalized by using the operator D_u in place of the ordinary differential operator.

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