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Solution of Certain Problems on the Unsaturated Flow of Liquids in a Porous Medium. (**)

1. - Introduction.

Presented here is a mathematical treatment of certain types of unsaturated flows of liquids (incompressible fluids) in a porous medium. The existence theory will be carried out by means of finite differences so that a viable numerical method for solving the relevant equations will be obtained as well.

The theory developed here governs such practical problems as those encountered in describing mathematically processes such as drainage, irrigation, movement of chemicals, liquid fertilizers and pollutants through soils, oil production, etc.. Take for example the case of a drain field. The field is initially saturated with water which will flow into neighboring ditches. The flow will continue until a certain, critical, residual saturation is reached at which time the flow ceases.

More generally, in a given domain a certain amount of liquid is concentrated at a relatively high pressure. As time progresses, the liquid will flow toward areas of lower pressure. The flow will continue as long as a sufficiently high saturation is maintained. When the saturation falls below a certain point, the flow will cease. Thus the problem: To determine the amount of liquid in any given point of the domain at any given time.

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To derive the mathematical equations, let $x = (x_1, \ldots, x_n)$ denote a point in n-dimensional Euclidean space, R_n, and $t \geqslant 0$ denote the time. Vector valued functions with values in R_n will be denoted by bold face, lower case Latin letters. The gradient of a function and the divergence of a vector valued function will be computed with respect to the x variables.

Under conditions of slow, steady (see e.g. BEAR, ZASLAVSKY and IRMAY [**3**], FULKS, GUENTHER and ROETMAN [**9**], POLUBARINOVA-KOCHINA [**19**] or SCHEIDEGGER [**23**], [**24**]), the flow is governed by Darcy's law

$$\boldsymbol{q} = -A(\operatorname{grad} p + \boldsymbol{f}) , \tag{1}$$

which relates the mass flux, $\boldsymbol{q}$, to the gradient of the pressure, p, and to the external body forces, $\boldsymbol{f}$, and the continuity equation

$$\frac{\partial}{\partial t} (\varrho \varphi S) + \operatorname{div} \boldsymbol{q} = g . \tag{2}$$

In equations (1) and (2), ϱ denotes the density of the liquid, φ denotes the porosity of the medium which is a measure of the pore volume available to the fluid, S denotes the saturation which gives the fraction of the pore space actually occupied by the fluid, $A = (a_{ij})$ denotes a positive definite, symmetric matrix which represents the resistance of the medium to the flow of the particular fluid in question and g denotes a function which arises in the event that absorption, pumping of the liquid out or into the domain, the depositing of material by the fluid on to the medium etc. takes place. A precise definition of these terms is given in [**9**] and a detailed physical discussion is given in [**3**], [**19**], [**23**], [**24**].

The known functions ϱ, φ, S, g, $\boldsymbol{f}$, a_{ij} are, in general, functions of pressure, temperature, position, time, etc., and their dependence on these quantities is complicated and difficult to measure. Consequently, it is necessary to make assumptions of both a physical and a mathematical nature to make the system (1) and (2) amenable to a mathematical treatment. The physical assumptions to be made are quite reasonable and include many of the situations met with in engineering problems. Desirable relaxations of these assumptions will be discussed in paragraph 5.

We now make the following physical assumptions:

(PA 1) Temperature dependence will be neglected.

(PA 2) $A = (a_{ij})$ is a positive definite, symmetric matrix which depends only on x.

(PA 3) ϱ is a positive constant.

(PA 4) f and g are functions of x, t and p.

(PA 5) The porosity φ depends upon x and p and satisfies the identity $0 < \varphi \leqslant 1$. As a function of p, φ is non-decreasing and for p sufficiently small, φ is independent of p.

Remark. In many applications φ is taken to be independent of p. However all media are slightly «spongy», i.e. φ always depends to some extent on p. This dependence is particularly pronounced, for example, in the case of oil reservoirs where very high pressures are present or in the case of loosely packed soils. Under these conditions, φ is usually assumed to be a linear or an exponential function of p in the range where φ depends upon p. See in this regard FULKS and GUENTHER [**8**], SCHEIDEGGER [**24**] and ŠČELKAČEV [**21**], [**22**].

(PA 6) S is a function of x and p and satisfies the inequality $0 \leqslant S \leqslant 1$. As a function of p, $S(x, p)$ is non-decreasing and for p sufficiently small, S is independent of p.

Combining now equations (1) and (2), we obtain

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial p}{\partial x_j}\right) + \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}(a_{ij}f_j) + g = \varrho \frac{\partial}{\partial t}(\varphi S)\,, \tag{3}$$

which in view of the assumptions (PA 1) ... (PA 6) is a nonlinear, partial differential equation in p alone which is of parabolic type but which for certain values of p degenerates to an elliptic differential equation.

Another type of nonlinear, degenerate parabolic equation arises in the study of gaseous flow in a porous medium. The appropriate equation is obtained by combining (1) and (2), taking $S \equiv 1$, $\varphi \equiv$ constant, and assuming an equation of state for ϱ (see LEIBENZON [**15**]). In one spatial dimension, this leads to the equation

$$\frac{\partial^2}{\partial x^2}\psi(x, t, p) = \frac{\partial p}{\partial t}\,. \tag{4}$$

Equation (4) was first studied in detail by OLEINIK, KALASHNIKOV and CHZOU YUI-LIN' [**17**]. Generalizations, extensions to higher dimensions and additional information on the behavior of the solutions to (4) have been obtained recently by ARONSON [**1**], [**2**], DUBINSKI [**6**], KALASHNIKOV [**12**] and others.

Not so extensively investigated are solutions to (3). Due to the special nature of the assumptions which are made concerning the function $\psi(x, t, p)$ in (4) and the functions φ and S in (3), the work of the above cited authors [**1**], [**2**], [**6**], [**12**], [**17**] is not applicable to (3). Moreover, their work is of a theoretical nature and finite difference methods are not discussed although DUBINSKI does use GALERKIN's method for constructing the solution. GAIPOVA **[10]** (see also her bibliography), in treating certain change of phase problems, has given a numerical method for approximating solutions to (3) when $a_{ij} = \delta_{ij}$, the KRONECKER delta, $\operatorname{div} \boldsymbol{f} \equiv g \equiv 0$ and φS depends on only on p. Convergence questions were not considered. In fact, existence and uniqueness questions for solutions to (3) seem to have been overlooked. The usual assumption that $(\partial/\partial p)(\varphi S)$ is bounded from below away from zero (see FRIEDMAN [**7**]) or at least that $(\partial/\partial p)(\varphi S) > 0$ (see KAMENOMOSTKAYA [**13**]) is not met here and so precludes a direct application of the standard theory.

In the sequel we shall obtain, after making certain normalizations and additional assumptions of a more mathematical nature, first the uniqueness of solutions to the first initial-boundary value problem for (3). A finite difference scheme will then be given for approximating the solutions and the convergence of the solutions of the difference scheme as the mesh is refined to the solution of the first initial-boundary value problem will be proven; thus, an existence theorem will also be obtained.

2. - Statement of the problem, mathematical assumptions, uniqueness.

Let Ω be a bounded domain (open connected set) in R_n and let σ be its boundary, which is assumed to be continuously differentiable. Let $\bar{\Omega} \equiv \Omega \cup \sigma$. Further, for any $t > 0$, define the sets

$$Q_t = \{(x, r) \mid x \in \Omega,\ 0 < r \leqslant t\}\ , \qquad S_t = \{(x, r) \mid x \in \sigma,\ 0 < r \leqslant t\}$$

and

$$\bar{Q}_t = Q_t \cup S_t \cup \{\bar{\Omega} \times (t = 0)\}\ .$$

Now let $T > 0$ be fixed but arbitrary. We seek a function $u = u(x, t)$ satisfying

$$\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) - \frac{\partial}{\partial t}\, b(x, u) = c(x, t)\, u + f(x, t)\ , \qquad (x, t) \in Q_T \tag{5}$$

$$u(x, 0) = \varphi(x)\ , \qquad x \in \Omega\ , \tag{6}$$

$$u(x, t) = 0\ , \qquad (x, t) \in S_T\ . \tag{7}$$

In the statement of the problem, the role of p in (3) is played by u, the role of $\varrho\varphi(x, u)\ S(x, u)$ is played by $b(x, u)$ and the role of $\operatorname{div} \boldsymbol{f} + g$ is played by

$c(x, t)\ u + f(x, t)$. This requirement on div f has physical ramifications which will be discussed in paragraph **5**.

In view of the subsequent smoothness assumptions on the known functions, we shall seek, not a « classical » solution to (5), (6), (7) as it stands, but instead a generalized solution. Motivated as usual by the requirement that a « classical » solution should also be a « generalized » solution and making use of the notation of IL'IN, KALASHNIKOV and OLEINIK [**11**], p. 90 et seq., we make the following

Definition 1. A function $u \in \overset{0}{W}{}^{1,0}(Q_T)$ is said to be a generalized solution (or simply a solution) to (5), (6), (7), if for all functions $\xi \in \overset{0}{W}{}^{1,1}(Q_T)$ with $\xi(x, T) = 0$, it satisfies the integral identity

$$(8) \qquad \int_{Q_T} \left[\sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial u}{\partial x_j} \frac{\partial \xi}{\partial x_i} - \frac{\partial \xi}{\partial t} b(x, u) \right] dx\, dt = \\ = \int_{\Omega} \xi(x, 0)\, b\big(x, \varphi(x)\big)\, dx - \int_{Q_T} [c(x, t)\, u\xi + f(x, t)\, \xi]\, dx\, dt\,,$$

where $dx = dx_1 \dots dx_n$.

To carry out the analysis, we shall make the following mathematical assumptions, which are of course partially motivated by the physical assumptions (PA 1) ... (PA 6) and the physical situations described above. Specifically we assume:

(MA 1) The functions $a_{ij}(x)$, $a_{ij}(x) = a_{ji}(x)$, $i, j = 1, \dots, n$ are defined and continuous on $\bar{\Omega}$. Further, there exist constants $a_0, a_1, a_1 \geqslant a_0 > 0$ such that for all $x \in \bar{\Omega}$, $\xi \in R_n$

$$a_0 \sum_{i=1}^{n} \xi_i^2 \leqslant \sum_{i,j=1}^{n} a_{ij}(x)\, \xi_i \xi_j \leqslant a_1 \sum_{i=1}^{n} \xi_i^2\,.$$

(MA 2) The function $b(x, u)$ is defined and continuous for all $x \in \bar{\Omega}$, $-\infty < u < \infty$, and $0 \leqslant b(x, u) \leqslant 1$. As a function of u, $b(x, u)$ is non-decreasing and satisfies a uniform LIPSCHITZ condition:

$$|b(x, u') - b(x, u'')| \leqslant M |u' - u''| \quad \text{for all} \quad x \in \bar{\Omega}\,, \quad -\infty < u', \quad u'' < \infty$$

where M is a fixed constant. Finally, for any $x \in \bar{\Omega}$, $b(x, u) = 0$ if $u \leqslant 0$ and $b(x, u)$ is strictly increasing in u for $u > 0$.

(MA 3) $c(x, t)$ and $f(x, t)$ are defined and continuous on $\bar{Q}_T$. Further, $c(x, t) \geqslant 0$ and as a function of t, $c(x, t)$ is continuously differentiable in $\bar{Q}_T$ and $\partial c/\partial t \geqslant 0$. Finally, $f(x, t) = 0$ for $(x, t) \in S_T$.

(MA 4) $\varphi(x)$ is defined and continuously differentiable for $x \in \bar{\Omega}$. Further, $\varphi(x) > 0$ for $x \in \Omega$ and $\varphi(x) = 0$ for $x \in \sigma$.

We now prove

Theorem 1. *There exists at most one solution (in the sense of Definition 1) to the problem* (5), (6), (7).

Proof. Suppose there existed two, $u^{(1)}$ and $u^{(2)}$. Let $w \equiv u^{(1)} - u^{(2)}$. Let t', $0 < t' \leqslant T$ be arbitrary and let $\xi(x, t) = \int_{t'}^{t} w(x, \tau)\,\mathrm{d}\tau$ for $t \leqslant t'$ and $\xi(x, t) = 0$ for $t > t'$. Then with this choice of ξ, it follows from (8) that

$$(9) \qquad \sum_{i,j=1}^{n} \int_{Q_{t'}} a_{ij}(x) \frac{\partial w}{\partial x_j} \left(\int_{t'}^{t} \frac{\partial w}{\partial x_i} \mathrm{d}\tau \right) \mathrm{d}x\,\mathrm{d}t - \int_{Q_{t'}} w[b(x, u^{(1)}) - b(x, u^{(2)})]\,\mathrm{d}x\,\mathrm{d}t =$$

$$= - \int_{Q_{t'}} c(x, t)\, w\Big(\int_{t'}^{t} w\,\mathrm{d}\tau \Big)\,\mathrm{d}x\,\mathrm{d}t\,.$$

Since b is non-decreasing in u,

$$(10) \qquad w[b(x, u^{(1)}) - b(x, u^{(2)})] \geqslant 0\,.$$

Further, by (MA 3)

$$(11) \quad \begin{cases} -\displaystyle\int_{Q_{t'}} cw\left(\int_{t'}^{t} w\,\mathrm{d}\tau \right) \mathrm{d}x\,\mathrm{d}t = -\tfrac{1}{2} \int_{Q_{t'}} c \frac{\partial}{\partial t} \left(\int_{t'}^{t} w\,\mathrm{d}\tau \right)^2 \mathrm{d}x\,\mathrm{d}t = \\ \qquad = \tfrac{1}{2} \displaystyle\int_{\Omega} c(x, 0) \left(\int_{t'}^{0} w\,\mathrm{d}\tau \right)^2 \mathrm{d}x + \tfrac{1}{2} \int_{Q_{t'}} \left[\frac{\partial}{\partial t} c \right] \left[\int_{t'}^{t} w\,\mathrm{d}\tau \right]^2 \mathrm{d}x\,\mathrm{d}t \geqslant 0\,. \end{cases}$$

Finally, by (MA 1)

$$(12) \quad \begin{cases} \displaystyle\sum_{i,j=1}^{n} \int_{Q_{t'}} a_{ij}(x) \frac{\partial w}{\partial x_j} \left(\int_{t'}^{t} \frac{\partial w}{\partial x_i} \mathrm{d}\tau \right) \mathrm{d}x\,\mathrm{d}t = \\ \qquad = \tfrac{1}{2} \displaystyle\sum_{i,j=1}^{n} \int_{Q_{t'}} a_{ij}(x) \frac{\partial}{\partial t} \left(\int_{t'}^{t} \frac{\partial w}{\partial x_j} \mathrm{d}\tau \int_{t'}^{t} \frac{\partial w}{\partial x_i} \mathrm{d}\tau \right) \mathrm{d}x\,\mathrm{d}t = \\ \qquad = -\tfrac{1}{2} \displaystyle\sum_{i,j=1}^{n} \int_{\Omega} a_{ij}(x) \int_{0}^{t} \frac{\partial w}{\partial x_j} \mathrm{d}\tau \int_{0}^{t} \frac{\partial w}{\partial x_i} \mathrm{d}\tau \leqslant -\frac{a_0}{2} \int_{\Omega} \sum_{i=1}^{n} \left(\int_{0}^{t'} \frac{\partial w}{\partial x_i} \mathrm{d}\tau \right)^2 \mathrm{d}x\,. \end{cases}$$

From the inequalities (9) ... (12) we conclude that $\int_0^{t'} \partial w/\partial x_i \, d\tau = 0$, $i = 1, \dots, n$ and since t' was arbitrary, $\partial w/\partial x_i = 0$ for $i = 1, \dots, n$. Thus, since $w \in \overset{0}{W}{}^{1,0}(Q_T)$, we conclude that $w \equiv 0$.

3. - The difference scheme.

Let $h > 0$ be a fixed number and $N \geqslant 1$ be a fixed integer and set $k = T/N$. Let $e_i = (0, 0, \dots, 1, \dots, 0)$, the one being in the i-th position, $i = 1, \dots, n$, denote the basis vectors in R_n. Construct the rectangular network of mesh points $R(h) = \{x \in R_n \,|\, x = (i_1 h, \dots, i_n h),\ i_1 = 0 \pm 1, \pm 2, \dots, l = 1, \dots, n\}$. Let $\bar{\Omega}(h) = R(h) \cap \bar{\Omega}$. If $x \in R(h)$, the points $x \pm h\, e_i$, $i = 1, \dots, n$ will be called the neighbors of x. The set of points in $\bar{\Omega}(h)$ which have at least one neighbor lying outside $\bar{\Omega}(h)$ will be denoted by $\sigma(h)$.

Set $\Omega(h) = \bar{\Omega}(h) - \sigma(h)$. Now let $t_m = mk$ and define the sets

$$Q(h) = \Omega(h) \times \{t = t_m \,|\, m = 1, \dots, N\}\,, \qquad S(h) = \sigma(h) \times \{t = t_m \,|\, m = 1, \dots, N\}$$

and

$$\bar{Q}(h) = Q(h) \cup S(h) \cup \{\bar{\Omega}(h) \times (t = 0)\}\,.$$

We now replace the problem (5), (6), (7) by a difference scheme. To this end, define the difference operators

$$D_i^+ U(x, t) = [U(x + he_i, t) - U(x, t)]/h\,, \qquad D_i^- U(x, t) = [U(x, t) - U(x - he_i, t)]/h\,,$$

$$D_t^+ U(x, t) = [U(x, t + k) - U(x, t)]/k\,, \qquad D_t^- U(x, t) = [U(x, t) - U(x, t - k)]/k\,.$$

We seek now a function, $U(x, t)$, defined on $\bar{Q}(h)$ satisfying

$$\text{(13)} \qquad \begin{cases} \sum\limits_{i,j=1} D_i^-\big(a_{ij}(x)\, D_j^+ U(x, t)\big) - D_t^- b\big(x, U(x, t)\big) = c(x, t)\, U(x, t) + f(x, t), \\ \hfill (x, t) \in Q(h)\,, \end{cases}$$

$$\text{(14)} \qquad U(x, t) = 0\,, \qquad (x, t) \in S(h)\,,$$

$$\text{(15)} \qquad U(x, 0) = \varphi(x)\,, \qquad x \in \Omega(h)\,,$$

where we take $U(x, 0)$ to be zero for $x \in \sigma(h)$. We shall also set $f(x, t)$ equal to zero for $(x, t) \in S(h)$ and $\varphi(x)$ equal to zero for $x \in \sigma(h)$ when dealing with the difference equations. Here and in the sequel we assume that h has been chosen so small, and certainly less than or equal to the diameter d of Ω, that $\Omega(h)$ is not empty. Then if Λ is the number of mesh points in $\Omega(h)$, (13) represents for each t_m, a nonlinear system of equations in Λ unknowns.

We first take up the question of the solvability of this system. We shall often be dealing with functions which vanish on $\sigma(h)$ or with the product of such a function and another function defined on $\bar{\Omega}(h)$. It will then often be convenient to think of these functions as being defined on all of $R(h)$ by simply assigning them the value zero in $R(h) - \Omega(h)$.

Let us first observe that if $V(x)$, $W(x)$ are defined on $R(h)$, then

$$W(x) D_j^- V(x) = D_j^-\big(W(x) V(x)\big) - V(x - he_j) D_j^+ W(x - he_j) . \tag{16}$$

Further, if $V(x) = 0$ on $R(h) - \Omega(h)$, then

$$\sum_{x \in R(h)} D_j^- V(x) = \sum_{x \in R(h)} D_j^+ V(x) = 0 . \tag{17}$$

From (16) and (17), it follows that if $V(x)$, $W(x)$ vanish on $R(h) - \Omega(h)$, then

$$\tag{18} \begin{cases} \displaystyle\sum_{x \in R(h)} V(x) \sum_{i,j=1}^{n} D_i^-\big(a_{ij}(x) D_j^+ W(x)\big) = \\ \displaystyle = - \sum_{x \in R(h)} \sum_{i,j=1}^{n} a_{ij}(x) D_j^+ V(x) D_i^+ W(x) . \end{cases}$$

Finally, if $V(x)$ vanishes on $R(h) - \Omega(h)$, then

$$\sum_{x \in R(h)} V^2(x) \leqslant 4d^2 \sum_{i=1}^{n} \sum_{x \in R(h)} [D_i^+ V(x)]^2 . \tag{19}$$

To prove (19), choose a fixed point $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n) \in \Omega(h)$ and sum the identity

$$D_i^+\big((x_i - \bar{x}_i) V^2(x)\big) = V^2(x + he_i) + (x_i - \bar{x}_i)\big(D_i^+ V(x)\big)\big(V(x + he_i) + V(x)\big)$$

over $R(h)$ and apply (17). Estimate $x_i - \bar{x}_i$ by d, apply the SCHWARZ inequality and the CAUCHY inequality ($2|ab| \leqslant \eta |a|^2 + (1/\eta)|b|^2$, $\eta > 0$ arbitrary) with $\eta = 4d$ to obtain (19).

Let us now introduce the linear space, $\mathscr{R}(h)$, of real valued functions which are defined on $R(h)$ and vanish on $R(h) - \Omega(h)$. Define for $V, W \in \mathscr{R}(h)$ the inner product $(V, W)_h = \sum_{x \in R(h)} V(x) W(x)$ and the norm $||V||_h^2 = (V, V)_h$. Let T be the mapping which takes $\mathscr{R}(h)$ into $\mathscr{R}(h)$ according to the formula

$$\text{(20)} \quad \begin{cases} TV(x) = -\sum_{i,j=1}^{n} D_i^-(a_{ij}(x) D_j^+ V(x)) + \frac{1}{k} b(x, V(x)) + c(x) V(x), & x \in \Omega(h)\,, \\ TV(x) = 0 \quad \text{for } x \in R(h) - \Omega(h) \end{cases}$$

Here $c(x)$ represents any one of the $c(x, t_m)$, $m = 1, \ldots, N$, of (13).

Theorem 2. *There exists precisely one solution $V \in \mathscr{R}(h)$ to the problem*

$$\text{(21)} \qquad TV = F\,,$$

where $F \in \mathscr{R}(h)$.
An immediate consequence of Theorem 2 is

Corollary 3. *There exists precisely one solution to the discrete problem* (13), (14), (15).

To prove Theorem 2, note first that by (18), (19), (MA 1), (MA 2) and (MA 3)

$$\text{(22)} \quad \begin{cases} (V-W, TV-TW) \geqslant \sum_{x \in R(h)} \sum_{i,j=1}^{n} a_{ij}(x) D_i^+(V(x) - W(x)) D_j^+(V(x) - W(x)) \geqslant \\ \qquad \geqslant a_0 \sum_{x \in R(h)} \sum_{i=1}^{n} [D_i^+(V(x) - W(x))]^2 \geqslant \frac{a_0}{4d^2} ||V - W||_h^2 . \end{cases}$$

Further, using (MA 1), (MA 2) and (MA 3), we can conclude the existence of a constant $K > 0$ such that

$$\text{(23)} \qquad ||TV - TW||_h \leqslant K(1/h^2 + 1/k + 1) ||V - W||_h .$$

Solving (21) is equivalent to finding a fixed point for the problem

$$T_\varrho V = V\,,$$

where $T_\varrho V = V - \varrho(TV - F)$ and $\varrho > O$ is a constant. But, from (22) and (23) it follows that T is contracting if one chooses $\varrho = a_0/4d^2 K(1/h^2 + 1/k + 1)$.

Remark. As pointed out in the introduction, Gaipova [**10**] has given a method for solving a more special form of the problem (13), (14), (15). A detailed discussion of the results and applications to various physical problems are also given. Furthermore, Gaipova's solution technique generalizes. See again [**10**]. The method used to prove Theorem 2 gives another technique for solving the numerical problem. The method is not new and is in fact standard when treating « monotone operators ». See [**18**]. However, other techniques are also available. In the event that $b(x, u)$ is differentiable with respect to u, numerous techniques are available.

See e.g. COLLATZ [**4**], RALL [**20**] and ORTEGA and RHEINBOLDT [**18**] and their bibliographies. In the event that $b(x, u)$ is not everywhere differentiable with respect to u, the fact that T satisfies (22) can be exploited. See in this regard KELLOGG [**14**] and particularly COLLATZ [**5**] and his bibliography.

4. - Existence theory.

We turn now to the investigation of the behavior of the solutions to (13), (14), (15) as $k, h \to 0$ with k/h^2 held constant. For these computations it will be necessary to make the following additional assumptions on the functions $c(x, t)$ and $f(x, t)$.

(MA 5) $f(x, t)$ is continuously differentiable with respect to t in $\bar{Q}_T$. Further, there exists a constant $a_2 \geqslant 0$ such that $\partial c/\partial t \leqslant a_2 c$.

First, it is necessary to obtain an a priori estimate on the solution $U(x, t)$ to the difference scheme (13), (14), (15). In these computations K will denote a constant which depends only on the bounds on the functions a_{ij}, c, $\partial c/\partial t$, f, $\partial f/\partial t$ and the constants mentioned in the assumptions (MA 1) ... (MA 5). Suppose then that $U(x, t)$ is the solutions to (13), (14), (15).

Following IL'IN, KALASHNIKOV and OLEINIK [**11**] page 100 et seq., we multiply equation (13) by $-\exp(-\theta t_m) D_t^- U(x, t_m)$ and sum over $x \in R(h)$, $m = 1, \ldots, m'$, where m' is any integer satisfying $1 \leqslant m' \leqslant N$ and $\theta > 0$ is a constant which will be chosen below. In the resulting equation, each term will be estimated separately. Using (18) and (MA 1) yields first

$$
(24)\quad \left\{
\begin{aligned}
&-\sum_{m=1}^{m'} \sum_{x \in R(h)} \sum_{i,j=1}^{n} \{D_i^-(a_{ij}(x) D_j^+ U(x, t_m))\} \exp(-\theta t_m) D_t^- U(x, t_m) = \\
&= \Sigma\Sigma\Sigma \exp(-\theta t_m)\, a_{ij}(x) D_j^+ U(x, t_m) D_i^+ D_t^- U(x, t_m) = \\
&= \tfrac{1}{2} \Sigma\Sigma\Sigma D_t^-(\exp(-\theta t_m)\, a_{ij}(x) D_j^+ U(x, t_m) D_i^+ U(x, t_m)) - \\
&\quad - \tfrac{1}{2} \Sigma\Sigma\Sigma(D_t^- \exp(-\theta t_m)\, a_{ij}(x) D_i^+ U(x, t_m) D_j^+ U(x, t_m) + \\
&\quad + \Sigma\Sigma\Sigma \exp(-\theta t_m)\, a_{ij}(x) \{\tfrac{1}{2} D_t^-(D_i^+ U(x, t_m) D_j^+ U(x, t_m)) - \\
&\qquad\qquad - D_i^+ U(x, t_{m-1}) D_t^- D_j^+ U(x, t_m)\} \geqslant \\
&\geqslant \frac{a_0}{2k} \exp(-\theta T) \sum_{x \in R(h)} \sum_{j=1}^{n} [D_j^+ U(x, t_{m'})]^2 - \frac{a_1}{2k} \sum_{x \in R(h)} \sum_{j=1}^{n} [D_j^+ \varphi(x)]^2 + \\
&\quad + \frac{a_0 \theta}{2} \exp(-\theta T) \sum_{m=1}^{m'} \sum_{x \in R(h)} \sum_{j=1}^{n} [D_j^+ U(x, t_m)]^2 ,
\end{aligned}
\right.
$$

since

$$\Sigma\Sigma\Sigma \exp(-\theta\, t_m)\, a_{ij}(x)\, \tfrac{1}{2} D_t(D_i^+ U(x, t_m)\, D_j^+ U(x, t_m)) -$$
$$- D_i^+ U(x, t_{m-1})\, D_t^- D_j^+ U(x, t_m) =$$
$$= \frac{k}{2} \Sigma\Sigma\Sigma \exp(-\theta t_m)\, a_{ij}(x)\, D_i^+ D_t^- U(x, t_m)\, D_j^+ D_t^- U(x, t_m) \geqslant 0\,.$$

Further, by (MA 2), $b(x, u)$ is non-decreasing in u, so

$$\exp(\theta t_m)\, D_t^- U(x, t_m)\, D_t^- b\big(x, U(x, t_m)\big) \geqslant 0\,. \tag{25}$$

Next, from the identities

$$\big((U, x, t_m) + U(x, t_{m-1})\big)\, c(x, t_m) \exp(-\theta t_m)\, D_t^- U(x, t_m) =$$
$$= D_t^-\big(U^2(x, t_m)\, c(x, t_m) \exp(-\theta t_m)\big) - U^2(x, t_{m-1})\, D_t^-\big(c(x, t_m) \exp(-\theta t_m)\big)\,,$$
$$\big(U(x, t_m) + U(x, t_{m-1})\big)\, c(x, t_m) \exp(-\theta t_m)\, D_t^- U(x, t) =$$
$$= 2U(x, t_m)\, c(x, t_m) \exp(-\theta t_m)\, D_t U(x, t_m) - c(x, t_m) \exp(-\theta t_m)\, k[D_t^- U(x, t_m)]^2\,,$$

follows

$$(26)\quad \left\{ \begin{aligned} &\sum_{m=1}^{m'} U(x, t_m)\, c(x, t_m) \exp(-\theta t_m)\, D_t U(x, t_m) = \\ &\qquad = \frac{1}{2k} U(x, t_{m'})^2\, c(x, t_{m'}) \exp(-\theta t_{m'}) - \\ &\qquad - \frac{1}{2k} \varphi^2(x)\, c(x, 0) - \sum_{m=1}^{m'} U^2(x, t_m)\, D_t^-\big(c(x, t_m) \exp(-\theta t_m)\big) + \\ &\qquad + k \sum_{m=1}^{m'} [D_t^- U(x, t_m)]^2\, c(x, t_m) \exp(-\theta t_m)\,. \end{aligned} \right.$$

Finally, an application of the identity (16) with D_t^- instead of D_j^- yields

$$(27)\quad \left\{ \begin{aligned} &\sum_{m=1}^{m'} [D_t^- U(x, t_m)]\, f(x, t_m) \exp(-\theta t_m) = \sum_{m=1}^{m'} D_t^-\big(\exp(-\theta t_m)\, U(x, t_m)\, f(x, t_m)\big) - \\ &- \sum_{m=1}^{m'} U(x, t_{m-1})\, D_t^+\big(f(x, t_{m-1}) \exp(-\theta t_{m-1})\big) = \\ &\qquad = \frac{1}{k} \exp(-\theta t_{m'})\, U(x, t_{m'})\, f(x, t_{m'}) - \\ &- \frac{1}{k} \varphi(x) f(x, 0) - \sum_{m=1}^{m'} U(x, t_{m-1})\, D_t^+\big(f(x, t_{m-1}) \exp(-\theta t_{m-1})\big)\,. \end{aligned} \right.$$

From (24) ... (27), (MA 1) ... (MA 5), (19), the fact that $\theta > 0$ is arbitrary, and several applications of CAUCHY's inequality $(2|ab| \leqslant \eta |a|^2 + 1/\eta |b|^2,\ \eta > 0$ arbitrary) with η suitably chosen, leads to the estimate

$$(28)\quad \left\{ \begin{aligned} &\sum_{m=1}^{m'} \sum_{x \in R(h)} \{ \sum_{i=1}^{n} [D_i^+ U(x, t_m)]^2 + [U(x, t_m)]^2 \} + \\ &+ \frac{1}{k} \sum_{x \in R(h)} \{ \sum_{i=1}^{n} [D_i^+ U(x, t_m)]^2 + [U(x, t_m)]^2 \} \leqslant \\ &\leqslant K \left[\sum_{m=1}^{n} \sum_{x \in R(h)} f^2(x, t_m) + \frac{1}{k} \sum_{x \in R(h)} (f^2(x, t_{m'}) + f^2(x, 0)) + \right. \\ &\left. + \frac{1}{k} \sum_{x \in R(h)} \{ \sum_{i=1}^{n} [D_i^+ \varphi(x)]^2 + \varphi^2(x) \} \right]. \end{aligned} \right.$$

Let now (x, t), $x \in R_n$, $0 \leqslant t \leqslant T$ be an arbitrary point. Let $x = (i, h, i_2 h, \ldots, i_n h)$ be a mesh point satisfying $(i_l - 1) h < x_l \leqslant i_l h$, $l = 1, \ldots, n$, and let t_m^- be such that $t_{m-1}^- < t \leqslant t_m^-$ and define the function $U(x, t; h)$ by setting $U(x, t; h) \equiv \equiv U(\bar{x}, \bar{t}_m)$. Finally, let $\lambda \equiv k/h^2$ be a fixed constant. Multiply (28) by $h^n k$ and replace $U(x, t)$ by $U(x, t; h)$. Then the summations over $x \in R(h)$, $m = 1, \ldots, m'$ on the left hand side in (28) become integrals over $R_n \times (0, t_{m'})$ of $U(x, t; h)$ (actually the integrals are over Qt_m, since the $U(x, t; h)$ vanish outside Q_{t_m}).

From this point on, the development parallels very closely that given in [**11**]. The functions $U(x, t; h)$ and $D_i U(x, t; h)$, $i = 1, \ldots, n$, are uniformly bounded in $L^2(Q_T)$ and for any t, in $L^2(\Omega)$. Consequently, we can find subsequences $\{U(x, t; h_p)\}_{p=1}^{\infty}$, $\{D_i U(x, t; h_p)\}_{p=1}^{\infty}$, $h_p > h_{p+1} \to 0$ as $p \to \infty$, λ constant, which converge weakly to functions $u(x, t) \in L^2(Q_T)$, $u_i(x, t) \in L^2(Q_T)$, $i = 1, \ldots, n$, and such that $b(x, U(x, t, h_p))$ converges weakly to a function $v(x, t) \in L^2(Q_T)$. As in [**11**], we can conclude that $u_i(x, t)$ is the generalyzed derivative of $u(x, t)$, i.e. $u_i(x, t) = \partial u(x, t)/\partial x_i$, $i = 1, \ldots, n$.

Further by a result of MAURIN [**16**], the subsequence can be so chosen that for fixed t, $\{U(x, t; h_p)\}_{p=1}^{\infty}$ converges in norm to $u(x, t)$ in $L^2(\Omega)$, which by (MA 2) allows us to conclude that $v(x, t) = b(x, u(x, t))$.

The limit function $u \in \overset{0}{W}{}^{1,0}(Q_T)$ satisfies the integral identity (8), and using the fact that $\varphi(x) \geqslant 0$, $u(x, 0) = \varphi(x)$. The assertion that u satisfies (8) is proven by multiplying (13) by a function $\xi \in \overset{0}{W}{}^{1,1}(Q_T)$, $\xi(x, T) = 0$ which is continuously differentiable, summing the resulting equality by parts, multiplying by $h_p^n k$ and then letting $h \to 0$ with λ held constant to obtain (8). We conclude, therefore, that the limit function $u(x, t)$ of $U(x, t; h_p)$ satisfies the problem (5), (6), (7) in the sense of Definition 1. Since from any subfamily of the family $U(x, t; h_p)$, we can extract a weakly convergent subsequence converging to a solution to (5), (6), (7) and since solutions to (5), (6), (7) are unique, we conclude that the whole family $U(x, t; h)$ converges weakly to the unique solution to (5), (6), (7). Thus, we obtain the following two theorems.

Theorem 3. *There exists a unique solution to the problem* (5), (6), (7) *in the sense of Definition* 1.

Theorem 4. *The solutions to the difference scheme* (13), (14), (15) *converge* (*weakly*) *to the solution of the problem* (5), (6), (7).

5. - Concluding remarks.

The theory developed above can be modified and extended in several ways. First, the continuity requirements on the $a_{ij}(x)$ could be replaced by the assumption that they be bounded and measurable. In that case, in the difference scheme (13), the $a_{ij}(x)$ would have to be replaced by suitable averages. Further, we could assume that the functions $\partial\varphi/\partial x_i$, $\partial c/\partial t$, $\partial f/\partial t$ exist as generalized derivatives, etc. Nonhomogeneous boundary conditions could also be treated. Further, in most applications, (a_{ij}) is a diagonal matrix, i.e. $(a_{ij}) = (a_i\delta_{ij})$. Assuming sufficient differentiability, local accuracy can be improved by replacing $(\partial/\partial x_i)(a_i(x)(\partial u/\partial x_i))$ by $D_i^-(a(x+he_i/2)\, D_i^+ U(x, t))$ and the analysis remains the same.

Of more substantial interest, however, is the elimination of the assumption (PA 1), a modification of the assumptions (PA 2), (PA 5), (PA 6) and the elimination of the assumption $\operatorname{div} \boldsymbol{f} + g$ is linear in the pressure (which is the effective assumption made in passing from (3) to (5)). This latter assumption rules out gravitational effects, since the exterior body force describing gravitational effects is $\varphi S \boldsymbol{g}$, where the acceleration $\boldsymbol{g}$ due to gravity has been assumed constant and this term should be included in the function $\boldsymbol{f}$.

Furthermore, the matrix (a_{ij}) is in general pressure dependent and this dependence has been explicitly ruled out. In the event that the fluid is made up of active chemicals or of radioactive elements, the composition of the fluid itself will be changing in time and possibly the fluid will be reacting with medium. Finally, it is desirable to allow $b(x, u)$ to have jump discontinuities in u so that certain change of phase problems could be treated by the same theory as well. Thus, the equation it would be desirable to treats is

$$(29) \qquad \sum_{i,j=1} \frac{\partial}{\partial x_i}\left(a_{ij}(x, t, u)\, \frac{\partial u}{\partial x_j}\right) = \frac{\partial}{\partial t}\, b(x, t, u) + f(x, t, u, \operatorname{grad} u)$$

in place of (5). The results of paragraphs 2-4 do not generalize directly to the case where (29) is used in phase of (5). Definition 1 can be easily generalized, but the proof of the uniqueness theorem fails. Natural difference schemes can be constructed which yield consistent approximations to (29), (6), (7); however, the theory of developed in paragraph 4 seems to be difficult to extend. From a physical standpoint, it would be very desirable to extend the results of paragraphs 2-4 to the problem (29), (6), (7).

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Summary.

This paper begins with a derivation of the equations governing the unsaturated flow of liquids in a porous medium. Uniqueness and existence theorems are proven. The proof of the existence theorem makes use of the method of finite differences so that a viable numerical method for approximating the solutions is also obtained.

Sommario.

Questo articolo comincia con la derivazione delle equazioni regolanti il flusso non saturato di liquidi nel medio poroso. Teoremi di unicità e di esistenza sono dimostrati. La dimostrazione di esistenza usa il metodo delle differenze finite; perciò si ottiene anche un metodo numerico per approssimare la soluzione.

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