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C. M. IOSHI and J. P. SINGHAL (*)

Operational Formulas Associated with a Class of Polynomials Unifying the Generalized Hermite and Laguerre Polynomials. (**)

1. - Introduction.

As long ago as 1941, BURCHNALL [2] made use of the operational formula

$$(1.1)\qquad (\mathrm{D}-2x)^n=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}H_{n-k}(x)\,\mathrm{D}^k\,,$$

where $\mathrm{D}=\mathrm{d}/\mathrm{d}x$, to prove the well-know relation

$$(1.2)\qquad H_{m+n}(x)=\sum_{k=0}^{\min(m,n)}(-2)^k\binom{m}{k}\binom{n}{k}k\,!\,H_{m-k}(x)\,H_{n-k}(x)\,,$$

due to NIELSEN [11]. Since then, much advance has been made toward the study of operational formulas associated with classical polynomials. For instance, GOULD and HOPPER [8] have established that

$$(1.3)\qquad \mathfrak{D}^n=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}H^r_{n-k}(x,\alpha,p)\,\mathrm{D}^k\,,$$

(*) Indirizzi degli A.A.: C. M. JOSHI, Department of Mathematics, Texas University, College Station, Texas, U.S.A.; J. P. SINGHAL, Department of Mathematics, University of Victoria, Victoria, B.C., Canada.

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where the symbol $\mathfrak{D}$ is defined by

$$\mathfrak{D} = \mathrm{D} - p r x^{r-1} + \alpha/x .$$

and satisfies the relation

$$x^n \mathfrak{D}^n = \prod_{j=0}^{n-1} (xD - prx^r + \alpha - j) ,$$

and

(1.4) $$H_n^r(x, \alpha, p) = (-1)^n x^{-\alpha} \exp(px^r) \mathrm{D}^n\{x^\alpha \exp(-px^r)\}$$

defines the elegant generalization of the HERMITE polynomials to which it reduces when $\alpha = 0, p = 1, r = 2$.

The relation (1.3) provides a generalization of the formula of BURCHNALL quoted above as well as of CARLITZ's formula [3]

(1.5) $$\prod_{j=1}^{n} (x\mathrm{D} - x + \alpha + j) = n! \sum_{k=0}^{n} \frac{x^k}{k!} L_{n-k}^{(\alpha+k)}(x) \mathrm{D}^k ,$$

for the LAGUERRE polynomials.

AL-SALAM [1], CHATTERJEA ([4], [5], [6]), DAS [7], R. P. SINGH [12] and many others have also studied the operational formulas for the classical polynomials and have either rederived already known results or obtained new ones.

In a preceding paper we [14] introduced a class of polynomials unifying the generalized HERMITE and LAGUERRE polynomials by means the RODRIGUES' formula

(1.6) $$J_n^{(\alpha)}(x, r, p, q) = C(q, n) x^{-\alpha} \exp(px^r) \cdot \mathrm{D}^n\{x^{\alpha+qn} \exp(-px^r)\} ,$$

where

$$C(q, n) = \frac{(-1)^{(n/2)(q-1)(q-2)}}{2^{(n/2)q(q-1)} (1)_{nq(2-q)}} ,$$

q being a non-negative integer, and

$$(a)_n = (a+1) \dots (a+n-1), \qquad n \geq 1, \ (a)_0 = 1 .$$

In the present paper we develop certain operational formulas for the generalized polynomial $J_n^{(\alpha)}(x, r, p, q)$, and make an attempt to unify the various results that appear in the literature.

2. - The operational formulas.

In proving the various results we shall make use of the differential operator $\delta = x(\mathrm{d}/\mathrm{d}x)$, which possesses the following interesting properties:

$$F(\delta)\{x^{\alpha} f(x)\} = x^{\alpha} F(\delta + \alpha) f(x) , \tag{2.1}$$

$$F(\delta) [\{\exp g(x)\} f(x)] = \{\exp g(x)\} F(\delta + x g') f(x) , \tag{2.2}$$

and

$$x^{n\alpha} F(\delta) F(\delta + \alpha) \dots F(\delta + (n-1) \alpha) = \{x^{\alpha} F(\delta)\}^{n} . \tag{2.3}$$

In view of the above mentioned formulas, it follows in a straight forward manner, that

$$\left\{\begin{aligned} &\mathrm{D}^{n} \{x^{\alpha+qn} \exp(-p x^{r}) Y\} = \\ &= x^{(q-1)n+\alpha} \exp(-p x^{r}) \prod_{j=1}^{n} (\delta + \alpha + (q-1) n - p r x^{r} + j) Y, \end{aligned}\right. \tag{2.4}$$

where Y is a sufficiently differentiable function of x.

On the other hand, we also have

$$\left\{\begin{aligned} &\mathrm{D}^{n} \{x^{\alpha+qn} \exp(-p x^{r}) \cdot Y\} = \\ &= 2^{(n/2)q(q-1)} x^{\alpha} \exp(-p x^{r}) \sum_{k=0}^{n} (-1)^{(n-k)(q-1)(q-2)/2} \binom{n}{k} \cdot \\ &\qquad \cdot (1)_{(n-k)q(2-q)} \left(\frac{x^{q}}{2^{q(q-1)/2}}\right)^{k} J_{n-k}^{(\alpha+qk)}(x, r, p, q)\, \mathrm{D}^{k} Y . \end{aligned}\right. \tag{2.5}$$

Therefore, a comparison of (2.4) and (2.5) readily yields the operational formula

$$\left\{\begin{aligned} &\prod_{j=1}^{n} (\delta + \alpha + (q-1) n - p r x^{r} + j) = \\ &= x^{(1-q)n} 2^{q(q-1)/2} \sum_{k=0}^{n} (-1)^{(n-k)(q-1)(q-2)/2} \binom{n}{k} (1)_{(n-k)q(2-q)} \cdot \\ &\qquad \cdot \left(\frac{x^{q}}{2^{q(q-1/2}}\right)^{k} J_{n-k}^{(\alpha+qk)}(x, r, p, q)\, \mathrm{D}^{k} . \end{aligned}\right. \tag{2.6}$$

Secondly on expressing

$$x^{-\alpha} \exp(p x^{r})\, \mathrm{D}^{n}\{x^{\alpha+qn} \exp(-p x^{r}) \cdot Y\}$$

in the form

$$x^{-\alpha-n}\exp(p\,x^r)\cdot\delta(\delta-1)\dots(\delta-n+1)\cdot\{x^{n-k}\,x^{\alpha+(q-1)n+k}\exp(-p\,x^r)\cdot Y\}$$

and using the relations (2.1), (2.2), (2.3) and (2.5) we are led to the formula

$$(2.7)\quad\begin{cases}\{x(\delta-k+1)\}^n\{x^{\alpha+(q-1)n+k}\exp(-p\,x^r)\cdot Y\}=\\ =x^{\alpha+k+n}\,2^{nq(q-1)/2}\sum_{s=0}^{n}(-1)^{(n-s)(q-1)(q-2)/2}\binom{n}{s}(1)_{(n-s)q(2-q)}\cdot\\ \cdot\left(\frac{x^q}{2^{q(q-1)/2}}\right)^s J_{n-s}^{(\alpha+qs)}(x,r,p,q)\,\mathrm{D}^s\,Y.\end{cases}$$

Next we observe that the recurrence relation (4.2, [14])

$$(x^q\mathrm{D}+\alpha\,x^{q-1}-p\,r\,x^{q+r-1})\,J_n^{(\alpha)}(x,r,p,q)=$$

$$=\frac{C(q,n)}{C(q,n+1)}\,J_{n+1}^{(\alpha-q)}(x,r,p,q)\,,$$

suggests the operational formula

$$(2.8)\qquad \mathfrak{D}_q^m\,J_n^{(\alpha)}(x,r,p,q)=\frac{C(q,n)}{C((q,m+n)}\,J_{m+n}^{(\alpha-mq)}(x,r,p,q)\,,$$

where

$$\mathfrak{D}_q\equiv x^q\mathrm{D}+\alpha\,x^{q-1}-p\,r\,x^{q+r-1}\,,$$

which corresponds to the formula (3.5, [8]) to which it reduces when $q=0$.

We also notice that when $q=0$, our formula (2.6) reduces to (1.3) referred to above and when $p=q=r=1$, we obtain the operational formula (1.5) due to CARLITZ [3]. Where as in the case $p=q=2$ and $r=-1$, our formula (2.6) assumes the form ([13], p. 129)

$$(2.9)\quad\prod_{j=1}^{n}(x\,\mathrm{D}+2x^{-1}+\alpha+2n-j+1)=x^{-n}\sum_{k=0}^{n}\binom{n}{k}2^{n-k}\,Y_{n-k}^{(\alpha+2k)}(x)\,\mathrm{D}^k\,,$$

where $Y_n^{(\alpha)}(x)$ are the generalized BESSEL polynomials of KRALL and FRINK [10]:

$$Y_n^{(\alpha)}(x)={}_2F_0[-n,n+\alpha+1;-;-\tfrac{1}{2}x]\,.$$

On the other hand, when $q = 1$, (2.7) yields the formula [9]

$$(2.10)\quad \begin{cases} \{x(x\mathrm{D} - k + 1)\}^n \{x^{\alpha+k} \exp(-p\,x^r) \cdot Y\} = \\ = x^{\alpha+k+n} \exp(-p\,x^r)\, n! \sum_{s=0}^{n} \frac{x^s}{s!} L_{n-s}^{(\alpha+s)}(x, r, p)\, \mathrm{D}^s Y, \end{cases}$$

where $L_n^{(\alpha)}(x, r, p)$ are the generalized LAGUERRE polynomials of SINGH and SRIVASTAVA [15]:

$$L_n^{(\alpha)}(x, r, p) = \frac{x^{-\alpha} \exp(p\,x^r)}{n!} \mathrm{D}^n \{x^{\alpha+n} \exp(-p\,x^r)\} .$$

If in addition to $q = 1$, we let $p = r = 1$ we shall obtain the formula of DAS [7] which reduces to the form

$$(2.11)\qquad \{(xD + 1)\}^n \{x^\alpha \exp(-x)\} = x^{\alpha+n} \exp(-x)\, n!\, L_n^{(\alpha)}(x) ,$$

due to AL-SALAM [1] when $k = 0$ and $Y = 1$.

It is also seen that when $q = k = 0$ and $Y = 1$ our formula (2.7) simplifies to

$$\{x(x\mathrm{D} + 1)\}^n \{x^{\alpha-n} \exp(-p\,x^r)\} = (-1)^n x^{\alpha+n} H_n^r(x, \alpha, p) .$$

3. - Some applications.

Setting $Y = 1$ in (2.6), we have

$$(3.1)\qquad \prod_{j=1}^{n} ((\delta + \alpha + (q-1)\,n - p\,r\,x^r + j) \cdot 1 = \frac{x^{(1-q)n}}{C(q, n)} J_n^{(\alpha)}(x, r, p, q) .$$

So that

$$\frac{x^{(1-q)(m+n)}}{C(q, m+n)} J_{m+n}^{(\alpha)}(x, r, p, q) = \prod_{j=1}^{m+n} \{\delta + \alpha + (q-1)(m+n) - p\,r\,x^r + j\} \cdot 1$$

$$= \prod_{j=1}^{m} \{\delta + \alpha + (q-1)(m+n) - p\,r\,x^r + j + n\} \cdot$$

$$\cdot \prod_{j=1}^{n} \{\delta + \alpha + (q-1)(m+n) - prx^r + j\} \cdot 1 =$$

$$= \frac{x^{(1-q)n}}{C(q, n)} \prod_{j=1}^{m} \{\delta + \alpha + n + (q-1)\,m - p\,r\,x^r + j\}\, J_n^{(\alpha+(q-1)m)}(x, r, p, q).$$

Therefore, in view of (2.5) we finally have

$$(3.2)\quad \left\{\begin{aligned} &\frac{C(q,\, m)\, C(q,\, n)}{C(q,\, m+n)}\, J^{(\alpha)}_{m+n}(x,\, r,\, p,\, q) = \\ &= \frac{1}{(1)_{mq(2-q)}} \sum_{k=0}^{m} (-1)^{k(q-1)(q-2)/2} \binom{m}{k} (1)_{(m-k)\,q(2-q)} \cdot \\ &\cdot \left(\frac{x^q}{2^{\frac{1}{2}q(q-1)}}\right)^k J^{(\alpha+kq+n)}_{m-k}(x,\, r,\, p,\, q)\ \mathrm{D}^k J^{(\alpha+(q-1)m)}_{n}(x,\, r,\, p,\, q). \end{aligned}\right.$$

If however, we reverse the order of the operators on the l.h.s. in (2.6) and proceed as above, we shall get an alternative formula

$$(3.3)\quad \left\{\begin{aligned} &\frac{C(q,\, m)\, C(q,\, n)}{C(q,\, m+n)}\, J^{(\alpha)}_{m+n}(x,\, r,\, p,\, q) = \\ &= \frac{1}{(1)_{mq(2-q)}} \sum_{k=0}^{m} (-1)^{k(q-1)(q-2)/2} \binom{m}{k} (1)_{(m-k)q(2-q)} \cdot \\ &\cdot \left(\frac{x^q}{2^{\frac{1}{2}q(q-1)}}\right)^k J^{(\alpha+qk)}_{m-k}(x,\, r,\, p,\, q)\ \mathrm{D}^k J^{(\alpha+qm)}_{n}(x,\, r,\, p,\, q). \end{aligned}\right.$$

A comparison of (3.2) and (3.3) leads us to the identity

$$(3.4)\quad \left\{\begin{aligned} &\sum_{k=0}^{m} (-1)^{k(q-1)(q-2)/2} \binom{m}{k} (1)_{(m-k)q(2-q)} \left(\frac{x^q}{2^{\frac{1}{2}q(q-1)}}\right)^k \cdot \\ &\cdot J^{(\alpha+n+qk)}_{m-k}(x,\, r,\, p,\, q)\ \mathrm{D}^k J^{(\alpha+(q-1)\,m)}_{n}(x,\, r,\, p,\, q) = \\ &= \sum_{k=0}^{m} (-1)^{k(q-1)(q-2)/2} \binom{m}{k} (1)_{(m-k)q(2-q)} \left(\frac{x^q}{2^{\frac{1}{2}q(q-1)}}\right)^k \cdot \\ &\cdot J^{(\alpha+qk)}_{m-k}(x,\, r,\, p,\, q)\ \mathrm{D}^k J^{(\alpha+qm)}_{n}(x,\, r,\, p,\, q)\,, \end{aligned}\right.$$

which in the special case $q = 0$, gives us

$$(5.3)\quad \left\{\begin{aligned} &\sum_{k=0}^{m} (-1)^k \binom{m}{k} H^r_{m-k}(x,\, \alpha,\, p)\ \mathrm{D}^k H^r_n(x,\, \alpha,\, p) = \\ &= \sum_{k=0}^{m} (-1)^k \binom{m}{k} H^r_{m-k}(x,\, \alpha+n,\, p)\ \mathrm{D}^k H^r_n(x,\, \alpha-m,\, p), \end{aligned}\right.$$

for the generalized HERMITE polynomials.

Next in (3.3) if we replace α by $\alpha - qm$, multiply both the sides by t^m and sum from $m = 0$ to $m = \infty$, we get

$$(3.6)\quad \begin{cases} \displaystyle\sum_{m=0}^{\infty} \frac{(1)_{(m+n)q(2-q)}}{m!}\, t^m J_{m+n}^{(\alpha-qm)}(x, r, p, q) = \\ \displaystyle = (1)_{nq\,(2-q)} J_n^{(\alpha)}\{x + A_q t x^q, r, p, q\} \sum_{m=0}^{\infty} \frac{(1)_{mq(2-q)}}{m!}\, t^m J_m^{(\alpha-qm)}(x, r, p, q)\,, \end{cases}$$

where A_q stands for

$$(-1)^{\frac{1}{2}(q-1)\,q-2)}\,(2)^{-\frac{1}{2}q(q-1)}\,.$$

Since it can be proved fairly easily that

$$(3.7)\quad \begin{cases} \displaystyle\sum_{m=0}^{\infty} \frac{(1)_{mq(2-q)}}{m!}\, t^m J_m^{(\alpha-qm)}(x, r, p, q) = \\ = \{1 + A_q t x^{q-1}\}^{\alpha} \exp\{p\, x^r - p\, x^r(1 + A_q t x^{q-1})^r\}. \end{cases}$$

(3.6) finally assumes the form

$$(3.8)\quad \begin{cases} \displaystyle\sum_{m=0}^{\infty} \frac{(1)_{(m+n)q(2-q)}}{m!}\, t^m J_{m+n}^{(\alpha-qm)}(x, r, p, q) = \\ = (1)_{nq(2-q)}\,(1 + A_q t x^{q-1})^{\alpha} \exp\{p\, x^r - p\, x^r(1 + A_q t x^{q-1})^r\}\,\cdot \\ \qquad\qquad \cdot\, J_n^{(\alpha)}(x + A_q t x^q, r, p, q)\,. \end{cases}$$

It is interesting to remark that the formula (3.8) admits a generalization of (5.3, [8]) to which it corresponds when $q = 0$, while $q = 1$ yields the relation

$$(3.9)\quad \begin{cases} \displaystyle\sum_{m=0}^{\infty} \binom{m+n}{m} t^m L_{m+n}^{(\alpha-m)}(x, r, p) = \\ = (1 + t)^{\alpha} \exp\{p\, x^r - p\, x^r(1 + t)^r\}\, L_n^{(\alpha)}\{x(1 + t), r, p\}\,, \end{cases}$$

and when $p = q = 2$ and $r = -1$, we obtain the generating relation

$$(3.10)\qquad \sum_{m=0}^{\infty} \frac{t^m}{m!}\, Y_{m+n}^{(\alpha-2m)}(x) = (1 + \tfrac{1}{2} x t)^{\alpha} \exp\frac{2t}{2 + xt} \cdot Y_n^{(\alpha)}\left\{x\left(1 + \frac{xt}{2}\right)\right\}\,,$$

for the BESSEL polynomials.

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Summary.

The present paper deals with certain operational formulas associated with a class of polynomials introduced by the authors [14] *which provides a unification of the various extensions of the Hermite and Laguerre polynomials given, for instance by Gould and Hopper* [8], *Singh and Srivastava* [15], *and many others referred to in* [14].

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