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Invertible Ideals in Orders. (**)

1. - Introduction.

This paper will consider the problem of classifying invertible ideals in orders. Sections **1** and **2** lay the groundwork and give the basic definitions. In Section **3** we give several conditions for invertibility and show that invertible ideals in orders behave much the same way as invertible ideals in DEDEKIND domains. In Section **4**, it is seen that in division algebras over the quotient field of a complete discrete valuation ring, invertibility is equivalent to projectivity. We consider the structure theory of invertible ideals in Section **5**. We end with two example illustrating points made in the paper.

2. - Preliminaries.

Let D be a DEDEKIND domain, K its quotient field and Σ a central simple K-algebra. A D-order in Σ is a subring Λ of Σ which is a finitely generated D-module, contains D and spans Σ over K, i.e., $\Lambda \otimes_D K \cong \Sigma$. It is clear that orders are two-sided Noetherian rings. A D-order Γ in Σ is maximal if it is not contained in any other D-order in Σ. It is known that maximal D-orders always exist [**1**].

We will assume that to say « an ideal in an order » means a fractionary ideal in the order, and assume that all ideals (one-sided or two-sided) are full in the sense that $I \otimes_D K \cong \Lambda \otimes_D K$. Finally, we assume that all Λ-modules considered are D-torsion free.

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Now let Λ be a D-order in Σ and let M be a left Λ-module which is D-torsion free. Define the left order of M in Σ, $\theta_l(M)$, by

$$\theta_l(M) = \{x \in \Sigma : xM \subseteq M\} .$$

If N is a right Λ-module, the right order of N in Σ, $\theta_r(N)$, is defined by

$$\theta_r(N) = \{x \in \Sigma : Nx \subseteq N\} .$$

It is clear that $\theta_l(M)$ is a D-order in Σ, $\theta_l(M) \supseteq \Lambda$, and that $\theta_l(M)$ is the largest D-order in Σ having M as a left module.

If I is a full left ideal of Λ, define $\theta_r(I)$ by

$$\theta_r(I) = \{x \in \Sigma : Ix \subseteq I\} .$$

Clearly, $\theta_r(I)$ is a D-order in Σ but will not contain Λ unless I is also a right Λ-ideal.

We will use the following notation:

$$(\Lambda : I)_r = \{x \in \Sigma : Ix \subseteq \Lambda\} ,$$

$$(\Lambda : I)_l = \{x \in \Sigma : xI \subseteq \Lambda\} .$$

Lemma 1. *Let I be a full left ideal of the D-order Λ. Then there are natural isomorphisms*

$$\theta_r(I) \cong \mathrm{Hom}_\Lambda (I, I)$$

and

$$(\Lambda : I)_r \cong \mathrm{Hom}_\Lambda (I, \Lambda) .$$

Proof. Let f be an element of $\mathrm{Hom}_\Lambda (I, \Lambda)$. Since $K \otimes_D I \cong K \otimes_D \Lambda \cong \Sigma$, let f' denote the unique extension of f to $K \otimes_D \mathrm{Hom}_\Lambda (I, \Lambda) \cong \mathrm{Hom}_\Sigma (\Sigma, \Sigma)$, ([**1**], lemma 2.4). Since $f(x) = xf'(1)$, $f'(1)$ is an element of $(\Lambda : I)_r$. Define

$$\mu : \mathrm{Hom}_\Lambda (I, \Lambda) \to (\Lambda : I)_r ,$$

by $\mu(f) = f'(1)$ for f in Hom_Λ (I, Λ); μ is easily seen to be a Λ-isomorphism.

The Λ-isomorphism

$$V : \mathrm{Hom}_\Lambda (I, I) \to \theta_r(I)$$

is defined similarly.

Lemma 2. *Let Λ be a D-order in Σ and let I be a full left Λ-ideal. In order that I be left Λ-projective, it is necessary and sufficient that there be a set of elements $\{\alpha_1, \ldots, \alpha_m\}$ in $(\Lambda: I)_r$ and a set of elements $\{x_1, \ldots, x_n\}$ in I such that*

$$\sum_{i=1}^{m} \alpha_i x_i = 1 .$$

Proof. From ([4], Proposition 3.1, p. 132) it is seen that I is Λ-projective if and only if there is a set $\{f_i\}$ in $\mathrm{Hom}_\Lambda(I, \Lambda)$ and a set $\{x_i\}$ in I such that $\sum_i f_i(x) x_i = x$ for all x in I and such that $f_i(x) = 0$ for all but a finite number of values of the index i. Let f_i denote the unique extension of f_i to $\mathrm{Hom}_\Sigma(\Sigma, \Sigma)$, for all i. Then using the isomorphism μ of Lemma 1 to identify $f_i'(1)$ with α_i in $(\Lambda: I)_r$ and using the fact that I is full, we have the desired conclusion.

Lemma 3. *Let $\Lambda, \Gamma \supseteq \Lambda$ be two D-orders in Σ and let M be a left Λ-module. If M is Λ-projective, then ΓM (the smallest left Γ-module containing M) is Γ-projective.*

Proof. It is sufficient to consider the case where M is free, and for this it is enough to look at the case where $M = \Lambda$. But then $\Gamma M = \Gamma$, so the result is clear.

3. - Invertible ideals.

Let I be a two-sided Λ-ideal. Say I is Λ-invertible if there is a two-sided Λ-module J such that $IJ = JI = \Lambda$.

First we remark that Example 11 will show that a full two-sided Λ-ideal may be both right and left Λ-projective but not invertible. However, we do have the following result.

Proposition 4. *Let D be a* Dedekind *domain, Λ a D-order in Σ and I a full two-sided Λ-ideal. Then I is Λ-invertible if and only if I is both right and left Λ-projective and $(\Lambda: I)_l = (\Lambda: I)_r$.*

Proof. Let I be Λ-invertible by J. Since $J \subseteq (\Lambda: I)_r$ and $(\Lambda: I)_r = JI(\Lambda: I)_r \subseteq J\Lambda = J$, we have that $J = (\Lambda: I)_r$. Similarly, $J = (\Lambda: l)_l$. Hence l is an element of $(\Lambda: I)_r I$ and of $I(\Lambda: I)_l$, so Lemma 2 and its counterpart for right ideals imply that I is both right and left Λ-projective.

Conversely, if I is left Λ-projective, we have from Lemma 2 that $(\Lambda: I)_r I$ contains 1, so $(\Lambda: I)_r I \supseteq \Lambda$. Also, $(\Lambda: I)_r I = (\Lambda: I)_l I \subseteq \Lambda$, so $(\Lambda: I)_r I = \Lambda$. In the same way, $I(\Lambda: I)_l = \Lambda$ and I is invertible by $(\Lambda: I)_r = (\Lambda: I)_l$.

When Λ is a D-order contained in Γ, a maximal D-order, Example 11 shows that an ideal I can be Γ-invertible but not Λ-invertible even if I is both right and left Λ-projective and $\theta_r(I) = \theta_l(I)$. We can, however, prove the following result.

Proposition 5. *Let Λ be a D-order, Γ a maximal D-order containing Λ and Σ a central simple K algebra. In order that I be Λ-invertible, it is necessary and sufficient that I be both right and left Λ-projective and $\Lambda = \theta_r(I) = \theta_l(I)$.*

Proof. Assume that I is Λ-invertible. Then Proposition 4 implies that I is right and left Λ-projective. Clearly, $\theta_r(I) \supseteq \Lambda$.

Let x be and element of $\theta_r(I)$. Then $Ix \subseteq I$ and $(\Lambda : I)_l Ix \subseteq (\Lambda : I)_l I$; i.e., $\Lambda x \subseteq \Lambda$, since I is invertible. Thus x is in Λ and $\theta_r(I) = \Lambda$. Similarly, $\theta_r(I) = \Lambda$.

Conversely, assume I is right projective. Define the map

$$\sigma\colon \operatorname{Hom}_\Lambda(I, \Lambda) \otimes_\Lambda I \to \operatorname{Hom}_\Lambda(I : I)$$

by setting $\sigma(f \otimes m)(n) = f(n)m$ for f in $\operatorname{Hom}_\Lambda(M, \Lambda)$ and m, n in Λ. Since I is right projective, σ is a two-sided $\operatorname{Hom}_\Lambda(I, I)$-isomorphism, ([**1**], proposition A-1). Set $I^* = (\Lambda : I)_l \cong \operatorname{Hom}_\Lambda(I, \Lambda)$ and consider the commutative diagram,

(1)
$$\begin{array}{ccc} I \otimes_\Lambda I^* & \xrightarrow{\sigma} & \operatorname{Hom}_\Lambda(I, I) \\ \downarrow & \nearrow_{\gamma} & \\ II^* & & \end{array}$$

where δ is the mapping defined by the universal mapping property of the tensor product, and γ is defined by $\gamma(w)z = wz$ for w in II^* and z in I. Since σ is an isomorphism, both δ and γ are isomorphisms. Thus we have the sequence of Λ-isomorphisms:

$$II^* \cong \operatorname{Hom}_\Lambda(I, I) \cong \theta_l(I) = \Lambda\,.$$

Nothe that $II^*I \subseteq I\Lambda \subseteq I$, so $II^* \subseteq \theta_l(I) = \Lambda$. Hence $II^* = \Lambda$. Similarly $I^*I = \Lambda$ and I is invertible.

Remark. If I is full two-sided for two orders Λ and Ω, then since $\theta_r(I)$ and $\theta_l(I)$ depend only on I and Σ, I can only be invertible in one of the orders Λ and Ω.

Lemma 6. *D is a Dedekind domain, I is full two-sided Λ-ideal. Let $I_{(P)}$, $\Lambda_{(P)}$, ... denote completion at a prime P of D. Then $I_{(P)}$ is $\Lambda_{(P)}$-invertible if and only if I is Λ-invertible.*

Proof. Since D is DEDEKIND domain, I is reflexive and $I = \bigcap_P I_{(P)}$, where the intersection runs over all the maximal primes P of D. Also, $I_{(P)} = \Lambda_{(P)}$ for all but a finite number of primes P. So if each $I_{(P)}$ is invertible by $I_{(P)}$ in $\Lambda_{(P)}$, set $J = \bigcap_P J_{(P)}$ and consider IJ. Clearly, $(IJ)_{(P)} = \Lambda_{(P)}$ for all P, so $\Lambda = \bigcap_P \Lambda_{(P)} = IJ$. Conversely, if I is invertible by J in Λ, then $(IJ)_{(P)} = I_{(P)} J_{(P)} = \Lambda_{(P)}$ for all P. Hence, $I_{(P)}$ is invertible in $\Lambda_{(P)}$ for all P.

Theorem 7. *Let D be a Dedekind domain with quotient field K and let Σ be a central simple K-algebra. Let Γ be an order in Σ. Then Γ is a maximal D-order if and only if every full two-sided ideal Γ is invertible.*

Proof. Let Γ be maximal: then Γ is left and right hereditary ([**1**], theorem 2.3), so every ideal I is both left and right projective. Since Γ is maximal, $\Gamma = \theta_r(I) = \theta_l(I)$ for any full two-sided Γ-ideal I. Therefore by Proposition 5 I is Γ-invertible.

Now assume that every full two-sided Γ-ideal is Γ-invertible. Then by Lemma 6 I is Γ-invertible if and only if $I_{(P)}$ is $\Gamma_{(P)}$-invertible for all maximal primes P of D. Since Γ is a maximal order if and only if $\Gamma_{(P)}$ is a maximal $D_{(P)}$ order in $\Sigma_{(P)}$ for all maximal primes P in D, it is sufficient to prove the theorem for a complete discrete rank one valuation ring.

Assume D is a complete discrete rank one valuation ring. Let I be a full two-sided Γ-ideal. If I is idempotent and J is its inverse, then

$$I = I\Lambda = IIJ = I^2 J = IJ = \Lambda .$$

Hence no nontrivial full two-sided Γ-ideal is idempotent. The JACOBSON radical N of Γ is two-sided so by Proposition 4 it is right and left projective. By ([**6**]: lemma 3.6), Γ is herediraty; so by ([**6**]: Theorem 1.7), Γ is a maximal order.

Corollary 8. *Σ is a central simple K-algebra. Γ is a maximal D-order in Σ. Every full two-sided Γ-ideal is a unique product of maximal full two-sided integral Γ-ideals. Further, the full two-sided ideals of Γ form an Abelian group.*

Remarks. The last statement of the corollary is known, ([**7**], theorem 7; p. 128). However, its proof is immediate from the proof of the first statement of the corollary so is included.

Proof of Corollary. Let I be a full two-sided integral ideal of Γ. Let M_1 be a maximal integral ideal such that $I \subseteq M_1$. By Theorem 7 M_1^{-1} exists so $I \subseteq IM_1^{-1} \subseteq \Gamma$. If $I = IM_1^{-1}$, then $\Gamma = I^{-1}I = I^{-1}IM_1^{-1}$; a contradiction. Hence $I \subset IM_1^{-1}$. If $IM_1^{-1} \neq \Gamma$, there is a maximal integral ideal M_2 such that $IM_1^{-1} \subseteq M_2$; as above we get $IM_1^{-1} \subseteq IM_1^{-1}M_2^{-1}$. Since Γ is Noetherian, this process must stop. Hence, there are maximal ideals $M_1, \ldots, M_r$ such that $IM_1^{-1}\ M_2^{-1} \ldots M_r^{-1} = \Gamma$; i.e., $I = M_r M_{r-1} \ldots M_1$.

Now let I be any ideal. Let $Q = \{x \in \Gamma : Ix \subseteq \Gamma\}$. Q and IQ are integral Γ-ideals such that $Q = M_1 \ldots M_r$ and $IQ = N_1 \ldots N_s$. Then $I = N_1 \ldots N_s$ $M_r^{-1} \ldots M_1^{-1}$.

To show the uniqueness, we will first show that the multiplication of maximal Γ-ideals is commutative. Clearly, in view of the fact that every ideal is a product of maximal ideals, multiplication of ideals is commutative.

Let M and N be maximal Γ-ideals. If $M = N$, then their multiplication commutes. If $M \neq N$, then by the first part of the proof $M \cap N = MI$ for some ideal I. Then $MI \subseteq N$, so $I \subseteq N$. Hence $M \cap N = MN$. Clearly, $M \cap N = NM$ follows by symmetry.

Now the classical proof of uniqueness can be applied, ([**10**], lemma 5, p. 272).

4. - Invertible ideals in division algebras.

Throughout this section D will be a complete discrete rank one valuation ring.

As was noted before, invertibility is not always equivalent to projectivity. However, when we are considering division algebras we have the following.

Theorem 9. *Let R be a finite dimensional division K-algebra. Let Λ be D-order in R, and let I be a full two-sided Λ-ideal. Then I is Λ-invertible if and only if I is both right and left Λ-projective.*

Proof. If I is Λ-invertible, Proposition 4 shows that I is both right and left Λ-projective.

If I is left Λ-projective then since D is complete, we can apply ([**5**], theorem 77.1) to obtain that I is Λ-isomorphic to a direct sum $\bigotimes_{i=1}^{n} \Lambda l_i$ with l_i an idempotent in Λ. However, since every idempotent in Λ is an idempotent in R, and since a division algebra has no nontrivial idempotents, $\Lambda l_i = \Lambda$. Thus

$I \cong \Lambda$, and there is a v in Λ such that $I = \Lambda v$. In the same way, using the fact that I is right Λ-projective, there is a u in Λ such that $I = u\Lambda$. Since $\theta_l(I) = \theta_l(\Lambda V) = \Lambda$ and $\theta_r(I) = \theta_r(u\Lambda) = \Lambda$, Proposition 5 implies that I is Λ-invertible.

5. - Structure theory.

This section will consider the structure theory of invertible ideals of orders when D is a complete discrete valuation ring. To this end, some background material will be required from [**3**]. For completeness we will give the essential definitions and results needed from [**3**] but will refer to that paper for the details.

We let K denote the quotient field of D and let π be a generator of the maximal ideal of D. Let Σ be a finite dimensional separable K-algebra. Let Λ denote a D-order in Σ and let $\bar{\Lambda} = \Lambda/\pi\Lambda$. Let

$$\bar{l} = \bar{l}_1 + \bar{l}_2 + \dots + \bar{l}_r$$

be a decomposition of $\bar{l}$ into primitive orthogonal idempotents. Then since D is a complete discrete valuation ring, there are primitive orthogonal idempotents $l_1, \dots, l_r$ in Λ which map to $\bar{l}_1, \dots, \bar{l}_r$ by the natural map $\Lambda \to \bar{\Lambda}$ and such that

$$l = l_1 + \dots + l_r .$$

We say that primitive orthogonal idempotents l_i and l_j in Λ are equivalent if

$$(l_i \Lambda l_j)(l_j \Lambda l_i) = l_i \Lambda l_i .$$

This is an equivalence relation, and we will write $l_i \sim l_j$ to denote the equivalence.

A D-order Λ in Σ is said to be reduced if its identity has a decomposition into primitive orthogonal idempotents, $1 = l_i + \dots + l_n$, such that no two distinct l_i and l_j are equivalent.

For the D-order Λ in Σ, let $f_1, \dots, f_k$ denote representatives of the distinct equivalence classes of equivalent idempotents. Set $f = f_1 + \dots + f_k$ and set $\tilde{\Lambda} = f\Lambda f$; then $\tilde{\Lambda}$ a reduced order in $f\Sigma f$. The map $I \to I = fIf = I \cap \tilde{\Lambda}$ of two-sided fractionary Λ-ideals to two-sided $\tilde{\Lambda}$-ideals is one-to-one and preserves products, sums and intersections. We will let m_i denote the number of distinct idempotents in the its equivalence class of equivalent idempotents. Example 12 will show that m_i is not necessarily equal to m_j.

The following theorem is the structure theorem for invertible ideals in reduced orders. The proof is essentially contained in the proof of Theorems 5 and 6 of [**3**].

Theorem 10. *Let* $\tilde{I}$ *be a full two-sided in the reduced D-order* $\tilde{\Lambda}$. *Then* $\tilde{I}$ *is* $\tilde{\Lambda}$*-invertible if and only if there is a permutation* σ *of the set* $\{1, 2, \dots, k\}$ *and an element* x *of* $\tilde{I}$ *such that* $\tilde{I} = x\tilde{\Lambda} = \tilde{\Lambda}x$, $x = x_1 + \dots + x_k$ *with* x_i *an element of* $f_i \tilde{I} f_{\sigma(i)}$ *and* $x_i f_{\sigma(i)} \Lambda f_j = f_i \tilde{I} f_j$. *Further, the inverse* $\tilde{J}$ *of* $\tilde{I}$ *is of the form* $\tilde{J} = y\tilde{\Lambda} = \tilde{\Lambda}y$, *where* y *is of the form* $y = y_1 + \dots + y_r$, y_i *in* $f_{\sigma(i)} \tilde{J} f_i$ *and* $x_i y_i = f_i$, $y_i x_i = f_{\sigma(i)}$*: In particular,* $\tilde{I}$ *and* $\tilde{J}$ *are* $\tilde{\Lambda}$*-free on one generator.*

Now we consider the case where Λ is not reduced. It is easy to see that if l_i, l_i', l_j, l_j' are idempotents in Λ such that $l_i \sim l_i'$ and $l_j \sim l_j'$, then $l_i \Lambda l_j \cong \cong l_i' \Lambda l_j'$ as D-modules. So we have I written as follows:

$$I = \sum_{\alpha,\beta} l_\alpha I l_\beta \cong \sum_{i,j} m_i m_j f_i I f_j$$

where $l_\alpha \sim f_i$, $l_\beta \sim f_j$ and the multiplier $m_i m_j$ signifies that there are $m_i m_j$ terms that are isomorphic to $f_i I f_j$: Further by Theorem 9,

$$I \cong \sum_{i,j} (x_i f_{\sigma(i)} \Lambda f_j) m_i m_j$$

where the x_i are the elements described in the statement of Theorem 10.

In view of Theorem 10, one might hope that I is Λ-cyclic. However, since the m_i are not necessarily equal, we do not have cyclicity (cfr. Examples 12).

When D is a DEDEKIND domain, one can reduce considerations to the case of a complete discrete valuation ring by localizing at the maximal primes of D. However, the structure of the invertible ideals now becomes rather complicated. There are two reason for this: first, as already noted, the m_i may be different; second, on localization we may obtain different sets of nonequivalent idempotents. For there reasons it appears to be best to give the structure in terms of local conditions.

6.- Examples.

In the examples which follow, A is a discrete rank one valuation ring, K is the quotient field of A, π is a generator of the maximal ideal of A, Σ_n is the K-algebra of $n \times n$ matrices with coefficients in K, Λ is an order in Σ. We will write $\Lambda = (A\pi^{r_{ij}})$, where this notation means that the elements of the (i, j) position are those elements in the A-ideal generated by $\pi^{r_{ij}}$.

Example 11. Set

$$\Gamma = \begin{bmatrix} A & A \\ A & A \end{bmatrix}, \qquad \Lambda = \begin{bmatrix} A & A \\ \pi A & A \end{bmatrix},$$

and

$$I = \begin{bmatrix} \pi A & \pi A \\ \pi A & \pi A \end{bmatrix}.$$

From ([**8**], theorem 2.6) Λ is hereditary, so I is both right and left Λ-projective. $I = \pi\Gamma = \Gamma\pi$, so I is Γ-invertible. Also, $\theta_r(I) = \theta_l(I) = \Gamma$, but

$$(\Lambda : I)_r = \begin{bmatrix} A & \pi^{-1}A \\ A & \pi^{-1}A \end{bmatrix} \quad \text{and} \quad (\Lambda : I)_l = \begin{bmatrix} \pi^{-1}A & \pi^{-1}A \\ A & A \end{bmatrix}.$$

Hence, by Proposition 5, I is not Λ-invertible. Therefore, there is a full two-sided Λ-ideal which is both right and left Λ-projective with $\theta_r(I) = \theta_l(I)$ but is not Λ-invertible.

Example 12. Set

$$\Lambda = \begin{bmatrix} A & A & A \\ \pi A & A & \pi A \\ A & A & A \end{bmatrix} \quad \text{and} \quad I = \begin{bmatrix} \pi A & A & \pi A \\ \pi A & \pi A & \pi A \\ \pi A & A & \pi A \end{bmatrix}.$$

Let l_{ii} denote the usual matrix idempotents. Then Λ is not reduced since $l_{11} \sim l_{33}$. I is Λ-invertible by

$$J = \begin{bmatrix} A & \pi^{-1}A & A \\ A & A & A \\ A & \pi^{-1}A & A \end{bmatrix}.$$

Set $f = l_{11} + l_{22}$ and $\tilde{\Lambda} = f\Lambda f$: i.e.,

$$\tilde{\Lambda} = \begin{bmatrix} A & A \\ \pi A & A \end{bmatrix}.$$

Further,

$$\tilde{I} = \begin{bmatrix} \pi A & A \\ \pi A & \pi A \end{bmatrix}.$$

Now set

$$x = \begin{bmatrix} 0 & 1 \\ \pi & 0 \end{bmatrix},$$

and then we have $\tilde{I} = \tilde{\Lambda}x = x\tilde{\Lambda}$. I is not Λ-cyclic. To see this, assume the contrary. On noting that $I^2 = \pi\Lambda$, we see that there are elements, a, b in Λ such that $x^2 = \pi a$, $xlx = \pi$. Then $(\det a)$ $(\det b) = 1$ and $(\det x)^2 = \pi^3(\det b)$, Let $(\det x) = \pi^s$. Then $2s = 3$, a contradiction.

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