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On Contraction Mappings. (**)

Introduction. - The contraction mapping principle of BANACH remains the most fruitful for proving the existence theorem in analysis. For this purpose extensions of the theorem are of continuing interest. In the present paper generalization of a few results of RAKOTCH [6] and JANOS [2] have been given. In Section 3, the notion of D-contraction (i.e. a mapping $T: X \to X$ of a metric space X into itself satisfying the condition $d(T_x, T_y) \leqslant \alpha\, d(x, T_x) + \beta\, d(y, T_y)$ for all x, $y \in X$ and $0 \leqslant \alpha + \beta \leqslant 1$) has been introduced. A theorem about D-contraction is also proven in the same section which generalizes the result of KANNAN [3]. Finally, a theorem on sequence of contraction mapping has been added which generalizes all the results on the sequence of contraction mappings which were based on BANACH Contraction Principle.

1. – Let X be a metric space and f a mapping of X into itself; f will be said to be globally contractive mapping if the condition

$$d\big(f(p), f(q)\big) < \lambda\ d(p, q)\ ,$$

with constant λ, $0 \leqslant \lambda \leqslant 1$, holds for every $p, q \in X, p \neq q$. A well known theorem of BANACH states:

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If X is a complete metric space and f is a globally contractive mapping of X into itself, then there exists a unique fixed point ξ such that $f(\xi) = \xi$.

1.1. – Definition. A mapping f of a metric space X into itself is said to be contractive if, for every two distinct points p and q in X,

$$d(f(p), f(q)) < d(p, q), \qquad \text{for all } p, q \in X,\ p \neq q .$$

A contractive mapping is clearly continuous, and if such a mapping has a fixed point, then this fixed point is unique.

1.2. – Definition. Denote by F the family of functions $\lambda(x, y)$ satisfying the following conditions:

(1) $\lambda(x. y) = \lambda(d(x, y))$, i.e. λ depends on the distance between x and y only.

(2) $0 \leqslant \lambda(d) < 1$ for every $d > 0$.

(3) $\lambda(d)$ is a monotonically decreasing function of d.

(4) $\operatorname{Sup} \lambda(x, y) = 1$.

1.3. – Definition. A metric space X will be said to be ε-chainable if for every $a, b \in X$ there exists an ε-chain, that is a finite set of points $a = x_0$, $x_1, \ldots, x_n = b$ (n may depend on both a and b) such that

$$d(x_{i-1}, x_i) < \varepsilon \qquad (i = 1, 2, \ldots, n).$$

Remark. Every connected metric space is well-linked but the converse is not true. For example, the set $\{(n, \tan x)\,|\,0 \leqslant x \leqslant \pi/2\} \cup \{(\pi/2, y)\,|\,y \geqslant 0\}$ is well-linked complete metric space but not connected. However, if X is compact, the converse is true. Further discussions about well-linked metric space may be found in CHOQNET [3], MATHEWS [11] and BERGE [1].

1.4 – Theorem. *If T is a contractive mapping of a complete ε-chainable metric space X into itself satisfying $0 < d(x, y) < \varepsilon \Rightarrow d(T(x), T(y)) \leqslant \lambda(x, y) \cdot d(x, y)$ $\forall x, y \in X$ and $(x, y) \in F$. Then T has a unique fixed point.*

Proof. Since (X, d) is ε-chainable we define, for every $x, y, \in X$,

$$d_\varepsilon(x, y) = \inf \sum_{i=1}^{n} d(x_{i-1}, x_i) ,$$

where the « infimum » is taken over ε-chains $x_0, x_1, \ldots, x_n$ joining $x = x_0$ and $y = x_n$.

Then d is a distance function on X satisfying

(i) $d(x, y) \leqslant d_\varepsilon(x, y)$,

(ii) $d(x, y) = d_\varepsilon(x, y) \quad$ for $\quad d(x, y) < \varepsilon$.

From (ii) is follows that a sequence $\{x_n\} \in X$ is a CAUCHY sequence with respect to d_ε if and only if it is a CAUCHY sequence with respect to d and is convergent with respect to d_ε if and only if it is convergent with respect to d. Since (X, d) is complete, therefore (X, d_ε) is also a complete metric space.

Moreover T is a contractive mapping with respect to d_ε. Given $x, y \in X$ and any ε-chain $x_0, x_1, \ldots, x_n$ with $x_0 = x$, $x_n = y$ we have

$$d(T_{x_{i-1}}, T_{x_i}) \leqslant \lambda(x_{i-1}, x_i)\, d(x_{i-1}, x_i) =$$

$$= \lambda\big(d(x_{i-1}, x_i)\big)\, d(x_{i-1}, x_i) < \lambda(\varepsilon)\,\varepsilon \qquad (i = 1, 2, \ldots, n)\,.$$

Thus, since $\lambda(\varepsilon) < 1$,

$$d(T_{x_{i-1}}, T_{x_i}) < \varepsilon \qquad (i = 1, 2, \ldots, n)\,.$$

Hence $T_{x_0}, T_{x_1}, \ldots, T_{x_n}$ is an ε-chain joining T_x and T_y and

$$d_\varepsilon(T_x, T_y) \leqslant \sum_{i=1}^{n} d(T_{x_{i-1}}, T_{x_i}) \leqslant \sum_{i=1}^{n} \lambda\big(d(x_{i-1}, x_i)\big)\, d(x_{i-1}, x_i)\,.$$

Since $x_0, x_1, \ldots, x_n$ is an arbitrary ε-chain, we have

$$d_\varepsilon(T_x, T_y) \leqslant \lambda\big(d_\varepsilon(x, y)\big)\, d_\varepsilon(x, y)\,.$$

Therefore, by the corollary to theorem 2 of RAKOTCH [**6**], T has a unique fixed point.

2.1. – Definition. Let $X = A_0$, $T(X) = A_1, \ldots, T^n(X) = A_n$ and introduce the functions $n(x)$ and $n(x, y)$ as follows

$$n(x) = \max\,\{n;\, x \in A_n\}, \qquad n(x, y) = \min\,\{n(x), n(y)\}\,.$$

2.2. – Theorem. *For $\lambda(x, y) \in F$ there exists a distance function d^* such that $d^*\big(T(x), T(y)\big) \leqslant \lambda\big(d(x, y), d^*(x, y)\big)$.*

Proof. By theorem 1 of JANOS [2] there exists a metric $d(x, y)$ with respect to which the mapping T is non-expansive. Let

$$\alpha(x, y) = \{\lambda(d(x, y))^{n(x, y)}, d(x, y)\} ,$$

$$A_n = T^n(X), \qquad T^{n+1}(X) = T(T^n(X)) .$$

Hence $x \in A_n$,

$$T(X) = A_{n+1} .$$

Let $i =$ (max. subscript for X). Then $i + 1 =$ (max. subscript for $T(X)$) . Let $j =$ (max. subscript for Y). Then $j + 1 =$ (max. subscript for $T(Y)$) . Thus $n(T(x), T(y)) = \min (i + 1, j + 1) = \min (i, j + 1) = n(x, y) + 1$,

$$\alpha(T(x), T(y)) \leqslant \lambda\, [d(x, y]^{n(T(x), T(y))}\, d(T_x, T_y) .$$

Now T is non-expansive, thus $d(T(x), T(y)) \leqslant d(x, y)$. Hence

$$(T(x), T(y)) \leqslant [\lambda(d(x, y))]^{n(x, y)+1}\, d(x, y) \leqslant \lambda(d(x, y))\, \alpha(x, y) .$$

The function $\alpha(x, y)$ is not in general a metric. However, a derived metric $d^*(x, y)$ can be defined as

$$d^*(x, y) = \inf \sum_{i=1}^{n} \alpha(x_i, x_{i-1}),$$

where the « infimum » is taken over all possible systems of elements x_1, x_2, ..., $x_n \in X$ such that $x = x_1$ and $x_{n+1} = y$.

From the definition of $d^*(x, y)$ it is clear that $d^*(x, y) \leqslant d(x, y)$. The same method as used by JANOS in [2] shows that $d^*(x, y)$ is a metric.

Now we have only to prove that

$$d^*(T(x), T(y)) \leqslant \lambda(d(x, y), d^*(x, y)) .$$

Let $\varepsilon > 0$ be given. From the definition of $d^*(x, y)$, there exists a representative of $d^*(x, y)$ in the form

$$d^*(x, y) = \inf \sum_{i=1}^{n} d(x_i,\ x_{i+1}) .$$

Thus

$$d^*(T(x),\, T(y)) \leqslant \inf \sum_{i=1}^{n} \alpha(T(x_i),\, T(x_{i+1})) \leqslant \inf \sum_{i=1}^{n} \lambda(d(x_i\,, x_{i+1}))\, \alpha(x_i\,, x_{i+1})$$

$$= \lambda \inf \sum_{i=1}^{n} \lambda(x_i\,, x_{i+1}) \inf \sum_{i=1}^{n} (x_i\,, x_{i+1}) = \lambda\,(d(x,\, y))\, d^*(x,\, y)\,.$$

Corollary. *When α is constant with $0 \leqslant \alpha \leqslant 1$ we get the result of* JANOS [2].

3.1. – Definition. A mapping $T\colon X \to X$ of a metric space X into itself is said to be D-contraction if it satisfies the following condition

$$d(T_x,\, T_y) \leqslant \alpha\, d(x,\, T_x) + \beta\, d(y,\, T_y) \quad \text{for all} \quad x, y \in X \text{ and } \alpha + \beta < 1\,.$$

3.2. – Theorem. *Every D-contraction of a complete metric space X into itself has a unique fixed point.*

Proof. Let a be an arbitrary point in X. Set $a_1 = T_a$, $Xa_2 = T_{a_1}$. So $a_2 = T_{a_1} = T_a^2$, and in general

$$Ta_{n-1} = T_a^n.$$

Claim. It can be shown that the sequence is CAUCHY. In fact we have

$$d(a_{n+1},\, a_n) = d(T_a^{n+1},\, T_{a-1}^n) \leqslant \alpha\, d(T_a^n,\, T_a^{n+1}) + \beta\, d(T_a^{n-1},\, T_a^n)$$

or

$$(1-\alpha)\;\; d(T_a^{n+1},\, T_a^n) < \beta\;\; d(T_a^{n-1},\, T_a^n)$$

or

$$d(T_a^{n+1},\, T_a^n) < \frac{\beta}{1-\alpha}\, d(T_a^{n-1},\, T_a^n)\,.$$

Now

$$d(T_a^n,\, T_a^{n-1}) < \alpha\, d(T_a^{n-1},\, T_a^n) + \beta\, d(T_a^{n-1},\, T_a^{n-2})$$

or

$$(1-\alpha)\, d(T_a^n, T_a^{n-1}) < \beta\, d(T_a^{n-1}, T_a^{n-2}),$$

i.e.

$$d(T_a^n, T_a^{n-1}) < \frac{\beta}{1-\alpha}\, d(T_a^{n-1}, T_a^{n-2}).$$

Hence

$$d(T_a^{n+1}, T_a^n) < \left(\frac{\beta}{1-\alpha}\right)^2 d(T_a^{n-1}, T_a^{n-2}).$$

Continuing this process:

$$d(T_a^{n+1}, T_a^n) < \left(\frac{\beta}{1-\alpha}\right)^n d(Ta, a).$$

Hence if n, m are integers satisfying $n, m \geqslant \delta$, then by triangle inequality, in general,

$$d(a_n, a_{n+p}) \leqslant d(a_n, a_{n+1}) + d(a_{n+1}, a_{n+2}) + \dots + d(a_{n+p-1}, a_{n+p})$$

$$\leqslant (r^n + r^{n+1} + \dots + r^{n+p-1})\, d(a, Ta),$$

where

$$r = \beta/(1-\alpha).$$

Since $0 < \alpha + \beta < 1$, $0 < r < 1$, therefore

$$d(\alpha_n, \alpha_{n+p}) \to 0 \quad \text{as } n \to \infty.$$

Therefore $\{a_n\}$ is a CAUCHY sequence.

Since X is complete, there exists a point $a_0 \in X$ which is the limit point of the sequence a_n, i.e. $\lim_{n\to\infty} a_n = a_0$.

Now we show that a_0 is a fixed point for T. By triangle inequality, we have

$$d(a_0, Ta_0) \leqslant d(a_0, a_n) + d(a_n, Ta_0) \leqslant d(a_0, a_n) + \alpha\, d(a_0, Ta_0) + \beta\, d(a_{n-1}, Ta_{n-1})$$

or

$$(1-\alpha)\, d(a_0, Ta_0) \leqslant d(a_0, a_n) + \beta\, d(a_n, a_{n-1}) .$$

Now for $\varepsilon > 0$ is arbitrary, then for sufficiently large n, we have

$$d(a_0, a_n) < \frac{(1-\alpha)\,\varepsilon}{1+\beta} \qquad \text{and} \qquad d(a_{n-1}, a_n) < \frac{(1-\alpha)\,\varepsilon}{1+\beta} .$$

Hence from (1) we have

$$d(a_0, Ta_0) \leqslant \frac{(1-\alpha)\,\varepsilon}{(1+\beta)(1-\alpha)} + \frac{(1-\alpha)\,\beta\varepsilon}{(1-\alpha)(1+\beta)} = \frac{\varepsilon}{1+\beta} + \frac{\beta\varepsilon}{1+\beta} = \varepsilon .$$

Corollary. *The theorem* 2 [3] *is a particular case of Theorem* 3.2 *by taking* $\alpha = \beta$.

3.3. – KANNAN ([3], [10]) has proven few theorems about fixed points by taking the following mapping $T\colon X \to X$ of a metric space X into itself satisfying the following condition

$$d(Tx, Ty) \leqslant \alpha\, [d(x, Tx) + d(y, Ty)]$$

for all $x, y \in X$ and $0 < \alpha < \frac{1}{2}$.

Lemma. *Every D-contraction satisfies the condition* (10).

Proof.

$$d(Tx, Ty) \leqslant \alpha\, d(x, Tx) + \beta\, d(y, Ty) ,$$

$$d(Ty, Tx) \leqslant \alpha\, d(y, Ty) + \beta\, d(x, Tx)$$

or

$$d(Tx, Ty) \leqslant \alpha\, d(y, Ty) + \beta\, d(x, Tx) ,$$

by symmetric property.

Adding (2) and (3) we have

$$2\, d(Tx, Ty) \leqslant (\alpha + \beta)\ [d(x, Tx) + d(y, Ty)] = \gamma\, [d(x, Tx) + d(y, Ty)],$$

where

$$\gamma = (\alpha + \beta)/2 < \tfrac{1}{2} .$$

4.1. – Theorem. *Suppose:*

(i) *X is a complete metric space.*

(ii) *$T_1, \ldots, T_n$ be a sequence of mappings with contraction coefficients α as in Definition* (1) *of Section* 3 *and with fixed points $x_1, x_2, \ldots, x_n$. Let $\lim_{n\to\infty} T_n = T$, with fixed point x.*

Then $\lim_{n\to\infty} x_n = x$.

Proof. Given that

$$\alpha(T_{nx}, T_{ny}) \leqslant \alpha(T_{(n-1)x}, T_{nx}) + \alpha(T_{(n-1)y}, T_{ny})$$

for all $x, y \in X$ and for $n = 1, 2, \ldots$.

Taking limit as $n \to \infty$ we get

$$d(Tx, Ty) \leqslant \alpha\, d(x, Tx) + \alpha\, d(y, Ty)\ .$$

Since T_n converges to T, therefore for given $\varepsilon > 0$ there exists an N such that $n \geqslant N$ implies $d(T_{nx}, T_x) \leqslant \varepsilon/(1+\alpha)$ where α is a contraction constant. Now for $n > N$,

$$d(x, x_n) = d(Tx, Tnx_n) \leqslant d(Tx, Tnx) + d(T_{nx}, T_n x_n) \leqslant$$

$$\leqslant d(Tx, T_{nx}) + \alpha d(x, T_{nx}) + \alpha d(X_n, T_n X_n) \leqslant d(T_x, T_{nx}) + \alpha\, d(T_x, T_{nx})\ .$$

Hence $d(x, x_n) \leqslant (1+\alpha)\, d(T_x, T_{nx})$ as the last factor $\to 0$. Thus, we have $d(x, x_n) < \varepsilon$ for $n \geqslant N$, so that $\lim_{n\to\infty} x_n = x$.

Suppose x' is another fixed point of T, then by above argument, $x' = \lim_{n\to\infty} x_n$. Hence T has only one fixed point.

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