Riv. Mat. Univ. Parma (3) 1 (1972), 91-96

ABRAR AHMAD KHAN (*)

On the Summability $|C, \alpha|_k$ of a Power Series. (**)

1. – Let $\sum a_n$ be a given infinite series with s_n as its n-th partial sum. We denote by $\{\sigma_n^\alpha\}$ and $\{t_n^\alpha\}$ the n-th (C, α) means of the sequence $\{s_n\}$ and $\{n\, a_n\}$ respectively. A series $\sum a_n$ is said to be summable $|C, \alpha|$ if $\sum_1^\infty |\sigma_n^\alpha - \sigma_{n-1}^\alpha| < \infty$ (1) and summable $|C, \alpha|_k$ $(\alpha > -1,\ k \geqslant 1)$ if

$$\sum_1^\infty n^{k-1} |\sigma_n^\alpha - \sigma_{n-1}^\alpha|^k < \infty \quad (2). \tag{1.1}$$

By virtue of a well known identity $t_n^\alpha = n\,(\sigma_n^\alpha - \sigma_{n-1}^\alpha)$, the condition (1.1) can also be written as

$$\sum_1^\infty |t_n^\alpha|^k / n < \infty . \tag{1.2}$$

2. – Concerning summability $|C, \alpha|$ of a power series, CHOW proved the following theorem.

Theorem A. If the radius of convergence of the power series

$$f(z) = \sum_{n=0}^\infty a_n z^n = \sum_{n=0}^\infty a_n r^n \exp(ni\theta) \tag{2.1}$$

(*) Indirizzo: Department of Applied Science, Faculty of Engineering, Aligarh Muslim University, Aligarh, India.

(**) Ricevuto: 12-X-1970.

(1) KOGBETLIANTZ [5], FEKETE [3].

(2) FLETT [4].

is unity and if

$$a_n = O(n^{\gamma}) \qquad (\gamma \geqslant -1),$$

then the series (2.1) is summable $|C, \alpha|$, for every $\alpha > \gamma + 1$, at every regular point of $f(z)$ on the unit circle (³).

Later on, in another paper he proved a number of results generalizing Theorem A. Among others, he proved the following theorems (⁴).

Theorem B. If $\sum |a_n|/n^{\alpha} < \infty$ $(\alpha > 0)$, and if $0 \leqslant \gamma < 1$ and $f'(z) = O(|\exp(i\theta_0) - z|^{-\gamma})$ in the neighbourhood of the point $\exp(i\theta_0)$, then the series $\sum a_n \exp(in\theta_0)$ is summable $|C, \alpha|$.

Theorem C. If $\sum |a_n|/n^{\alpha} < \infty$ $(\alpha > 0)$, and $|f'(z)| < \chi(\theta)$ in an arc (γ, β), where $\chi(\theta)$ is integrable in LEBESGUE's sense in (γ, β), and if the function

$$\Phi(\theta) = \frac{1}{\theta - \theta_0} \int_{\theta_0}^{\theta} \chi(\varphi)\, d\varphi \qquad (\gamma < \theta_0 < \beta)$$

is integrable in LEBESGUE's sense in (γ, β), then the series $\sum a_n \exp(in\theta_0)$ is summable $|C, \alpha|$.

3. – The object of this paper is to generalize the Theorems B and C mentioned above, by considering the summability $|C, \alpha|_k$ $(k \geqslant 1)$. Our theorems are as follows:

Theorem 1. *If* $\sum |a_n|^k / n^{1+k\alpha-k} < \infty$ $(k \geqslant 1,\ \alpha > 0,\ 0 \leqslant \gamma < 1)$ *and* $f'(z) = O(|\exp(i\theta_0) - z|^{-\gamma})$ *in the neighbourhood of the point* $\exp(i\theta_0)$, *then the series* $\sum a_n \exp(in\theta_0)$ *is summable* $|C, \alpha|_k$.

Theorem 2. *If* $\sum |a_n|^k / n^{1+k\alpha-k} < \infty$ $(k \geqslant 1,\ \alpha > 0)$ *and* $|f'(z)| < \chi(\theta)$, *in an arc* (γ, β) *where* $\chi(\theta)$ *is integrable in Lebesgue's sense in* (γ, β) *and if*

$$\int_{\gamma}^{\beta} \Phi^k(\theta)\, d\theta < \infty \qquad (k \geqslant 1),$$

(³) CHOW [1].
(⁴) CHOW [2].

where

$$\Phi(\theta) = \frac{1}{\theta - \theta_0} \int_{\theta_0}^{\theta} \chi(\varphi)\, d\varphi \qquad (\gamma < \theta_0 < \beta)\,,$$

then the series $\sum a_n \exp(in\theta_0)$ *is summable* $|C, \alpha|_k$.

4. – For the proof of the above theorems we require the following

Lemma (5). Let θ be a point within an arc (γ, β) on the unit circle. If $\alpha > 0$ and

$$\sum n^{k-1-k\alpha} |a_n|^k < \infty \qquad (k \geqslant 1)\,,$$

the necessary and sufficient condition that the series $\sum a_n \exp(ni\theta)$ should be summable $|C, \alpha|_k$ is that

$$\sum n^{-1-k\alpha} \left| \int_{\gamma}^{\beta} f'(z) \left\{(\exp i\theta) - z)^{-\alpha} + c_0 + c_1 z\right\} z^{-n}\, dz \right|^k < \infty\,,$$

where $|z| = r = 1 - (1/n)$ and c_0 and c_1 are functions of θ but independent of n.

5. – Proof of Theorem 1. We assume that $\theta_0 = 0$ and $\alpha + \gamma > 1$ as this does not affect the generality. Let $\gamma = -\delta$, $\beta = \delta$, where δ is a small positive number. Then by above Lemma it is sufficient to prove that

$$\sum_{1}^{\infty} n^{-1-k\alpha} \left| \int_{-\delta}^{\delta} f'(z) \left\{(1-z)^{-\alpha} + c_0 + c_1 z\right\} z^{-n}\, dz \right|^k < \infty\,.$$

Now, for $|z| = r = 1 - (1/n)$,

$$\sum_{1}^{\infty} n^{-1-k\alpha} \left| \int_{-\delta}^{\delta} f'(z) \left\{c_0 + c_1 z\right\} z^{-n}\, dz \right|^k \leqslant$$

$$\leqslant C \sum_{n=1}^{\infty} n^{-1-k\alpha} \left\{ \int_{-\delta}^{\delta} |1 - r\exp(i\varphi)|^{-\gamma} \left\{|c_0| + |c_1| r\right\} r^{-n+1}\, d\varphi \right\}^k \text{ (6)}$$

$$\leqslant C \sum_{n=1}^{\infty} n^{-1-k\alpha} \left\{ \int_{-\delta}^{\delta} |1 - r\exp(i\varphi)|^{-\gamma}\, d\varphi \right\}^k \leqslant C \sum_{n=1}^{\infty} n^{-1-k\alpha} < \infty \qquad (\alpha > 0,\ k \geqslant 1)\,.$$

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(5) SINGH [6].

(6) Where C is a constant not necessarily the same at each occurrence.

Also

$$\sum_{1}^{\infty} n^{-1-k\alpha} \left| \int_{-\delta}^{\delta} f'(z)(1-z)^{-\alpha} z^{-n}\, dz \right|^k \leqslant$$

$$\leqslant C \sum_{1}^{\infty} n^{-1-k\alpha} \left\{ \int_{-\delta}^{\delta} |1 - r \exp(i\varphi)|^{-\alpha-\gamma}\, d\varphi \right\}^k$$

$$\leqslant C \sum_{1}^{\infty} n^{-1-k\alpha} n^{(\gamma+\alpha-1)k} = C \sum_{1}^{\infty} n^{k\gamma-1-k} < \infty \qquad (\gamma < 1)\,.$$

This completes the proof of Theorem 1.

6. – Proof of Theorem 2. Suppose $\theta_0 = 0$, so that $\gamma < 0 < \beta$. As in the proof of Theorem 1, we have

$$\sum_{n=1}^{\infty} n^{-1-k\alpha} \left| \int_{\gamma}^{\beta} f'(z)(c_0 + c_1 z) z^{-n}\, dz \right|^k \leqslant C \sum_{n=1}^{\infty} n^{-1-k\alpha} \left\{ \int_{\gamma}^{\beta} \chi(\theta)\, d\theta \right\}^k < \infty\,,$$

and

$$\sum_{n=1}^{\infty} n^{-1-k\alpha} \left| \int_{\gamma}^{\beta} f'(z)(1-z)^{-\alpha} z^{-n}\, dz \right|^k \leqslant C \sum_{n=1}^{\infty} n^{-1-k\alpha} \left\{ \int_{\gamma}^{\beta} \frac{\chi(\theta)}{\{(1-r)^2 + \theta^2\}^{\alpha/2}}\, d\theta \right\}^k \leqslant$$

$$\leqslant C \sum_{n=1}^{\infty} n^{-1} \left\{ \int_{\gamma}^{\beta} \frac{\chi(\theta)}{(1+n^2\theta^2)^{\alpha/2}}\, d\theta \right\}^k = C \sum_{n=1}^{\infty} \frac{1}{n} \left\{ \int_{0}^{\beta} \frac{\chi(\theta)}{(1+n^2\theta^2)^{\alpha/2}}\, d\theta + \int_{\gamma}^{0} \frac{\chi(\theta)}{(1+n^2\theta^2)^{\alpha/2}}\, d\theta \right\}^k$$

$$\leqslant C \sum_{n=1}^{\infty} \frac{1}{n} \left(\int_{0}^{\beta} \frac{\chi(\theta)}{(1+n^2\theta^2)^{\alpha/2}}\, d\theta \right)^k + C \sum_{n=1}^{\infty} \frac{1}{n} \left(\int_{\gamma}^{0} \frac{\chi(\theta)}{(1+n^2\theta^2)^{\alpha/2}}\, d\theta \right)^k$$

$$= J_1 + J_2\,, \qquad \text{say.}$$

$$\int_{0}^{\beta} \frac{\chi(\theta)}{(1+n^2\theta^2)^{\alpha/2}}\, d\theta = \left[\frac{\theta\, \Phi(\theta)}{(1+n^2\theta^2)^{\alpha/2}} \right]_{0}^{\beta} + \alpha \int_{0}^{\beta} \frac{n^2\theta^2\, \Phi(\theta)}{(1+n^2\theta^2)^{(\alpha/2)+1}}\, d\theta$$

$$\leqslant C n^{-\alpha} + C \int_{0}^{\beta} \frac{n^2\theta^2\, \Phi(\theta)}{(1+n^2\theta^2)^{(\alpha/2)+1}}\, d\theta\,.$$

Therefore

$$J_1 \leqslant C \sum_{n=1}^{\infty} \frac{1}{n} \left\{ n^{-\alpha} + \int_0^\beta \frac{n^2 \theta^2 \Phi(\theta)}{(1+n^2\theta^2)^{(\alpha/2)+1}} \, d\theta \right\}^k$$

$$\leqslant C \sum_{n=1}^{\infty} n^{-1-k\alpha} + C \sum_{n=1}^{\infty} \frac{1}{n} \left\{ \int_0^\beta \frac{n^2 \theta^2 \Phi(\theta)}{(1+n^2\theta^2)^{(\alpha/2)+1}} \, d\theta \right\}^k,$$

$$C \sum_{n=1}^{\infty} \frac{1}{n} \left\{ \int_0^\beta \frac{n^2 \theta^2 \Phi(\theta)}{(1+n^2\theta^2)^{(\alpha/2)+1}} \, d\theta \right\}^k \leqslant C \sum_{n=1}^{\infty} n^{2k-1} \left\{ \int_0^\beta \frac{\theta^{2k} \Phi^k(\theta)}{(1+n^2\theta^2)^{(2^{-1}\alpha+1)k}} \, d\theta \right\} \left\{ \int_0^\beta d\theta \right\}^{k/k'}$$

$$\leqslant C \sum_{n=1}^{\infty} n^{2k-1} \int_0^\beta \frac{\theta^{2k} \Phi^k(\theta)}{(1+n^2\theta^2)^{(2^{-1}\alpha+1)k}} \, d\theta = C \int_0^\beta \Phi^k(\theta) \sum_{n=1}^{\infty} \frac{n^{2k-1}\theta^{2k}}{(1+n^2\theta^2)^{(2^{-1}\alpha+1)k}} \, d\theta$$

$$= C \int_0^\beta \Phi^k(\theta) \Big(\sum_{n \leqslant \theta^{-1}} + \sum_{n > \theta^{-1}} \Big) \, d\theta$$

$$\leqslant C \int_0^\beta \Phi^k(\theta) \Big(\sum_{n \leqslant \theta^{-1}} n^{2k-1}\theta^{2k} \Big) \, d\theta + C \int^\beta \Phi^k(\theta) \left(\sum_{n > \theta^{-1}} \frac{n^{2k-1}\theta^{2k}}{n^{2k+\alpha k}\, \theta^{2k+\alpha k}} \right) d\theta$$

$$\leqslant C \int_0^\beta \Phi^k(\theta) \Big(\sum_{n \leqslant \theta^{-1}} \theta \Big) \, d\theta + C \int_0^\beta \Phi^k(\theta) \sum_{n > \theta^{-1}} n^{-\alpha k-1} \theta^{-\alpha k} \, d\theta$$

$$\leqslant C \int_0^\beta \Phi^k(\theta) \, d\theta + C \int_0^\beta \Phi^k(\theta) \, d\theta < \infty .$$

Hence $J_1 = O(1)$. Similarly $J_2 = O(1)$.

This completes the proof of Theorem 2.

I wish to thank Dr. S. M. Mazhar for his help in the preparation of this paper.

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