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CHARLES A. COPPIN and JOSEPH F. VANCE (*)

On a Generalized Riemann-Stieltjes Integral. (**)

This paper deals with a special case of HENSTOCK's generalized RIEMANN integral [2]. In 1967, VANCE [4] showed how this concept could be used in characterizing certain bounded linear functionals; and, in 1968, COPPIN [1] developed some additional results.

1. - Introduction.

Throughout this paper, $[a, b]$ denotes a closed number interval and Δ denotes the set of all subsets of $[a, b]$ whose closure is $[a, b]$ and which contains a and b. If A is a number set with an upper bound, then a number a is said to be a right end point of A if and only if a belongs to A and $a = \sup A$. Left end point is defined similarly, and it should be obvious how end point is defined. Two sets of numbers, A and B, are said to be nonoverlapping if and only if $A \cap B = \emptyset$ or $A \cap B$ is a singleton. A nonempty collection is said to be nonoverlapping if and if each two distinct members of the collection is nonoverlapping. All functions considered here are real-valued of sets only have domain including $[a, b]$, and are bounded on $[a, b]$. Also, if M is a set and f is a function then

$$f \mid M = \{(x, f(x)) \mid x \in M \text{ and } x \text{ is in the domain of } f\}.$$

(*) Indirizzo degli Autori: C. A. COPPIN, Department of Mathematics, University of Dallas, Irving, Texas, U.S.A.; J. F. VANCE, Department of Mathematics, St. Mary's University, San Antonio, Texas, U.S.A..

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Definition. If M is a member of Δ, then D is said to be a partition of M if and only if D is a finite collection of nonoverlapping subsets of M whose union is M and each member of D has two end points. $E(D)$ will denote the set of end points of members of D.

Definition. If M is a member of Δ and D is a partition of M, then D' is said to be a refinement of D if and only if D' is a partition of M and $E(D)\subset E(D')$.

Definition. If D is a nonempty collection of nonempty sets, then δ is said to be a choice function on D if and only if δ is a function with domain including D such that $\delta(d)\in d$ for each d in D.

Definition. If D is a subset of a partition of a member of Δ, δ is a choice function on D, and f and g are functions with domains including $\cup D$, then

$$\Sigma(f, g, D, \delta) = \sum_{i=1}^{n} f\big(\delta([x_{i-1}, x_i])\big)\cdot[g(x_i)-g(x_{i-1})]\,,$$

where $x_0, x_1, \ldots, x_n$ are the end points of the members of D and $x_0 < x_1 < \ldots < x_n$.

Definition. Suppose that M is a member of Δ and f and g are functions with domain including M. Then f is said to be g-integrable on M if and only if there exists a number W (denoted by $\int_M f\,dg$) such that for each positive number ε, there is a partition D of M such that

$$|W-\Sigma(f, g, D', \delta)|<\varepsilon$$

for each refinement D' of D and each choice function δ on D'.

2. - Result.

The following theorem, the proof of which is analogous to a proof of Theorem 10.2, page 49, in [**3**], will prove to be useful.

Theorem 1. *Suppose that f and g are functions with domain $[a, b]$, and M is a member of Δ. Then the following two statements are equivalent:*

(a) *f is g-integrable on M.*

(b) *If* $\varepsilon > 0$, *there exists a partition* D *of* M *such that* $|\Sigma(f, g, D, \delta) - \Sigma(f, g, D', \delta')| < \varepsilon$ *for each choice function* δ *on* D, *each refinement* D' *of* D, *and each choice function* δ' *on* D'.

Theorem 2. *Suppose that* f *and* g *are functions with domain* $[a, b]$, *and* M *is a member of* Δ. *Then the following two statements are equivalent*:

(a) f *is* g-*integrable on* M, *and* $f|M$ *and* $g|M$ *have no common points of discontinuity*.

(b) *If* M' *is a countable member of* Δ *and a subset of* M, *then* f *is* g-*integrable on* M'.

Proof. Suppose that f and g are functions with domain $[a, b]$, and M is a member of Δ.

We will first prove that (b) implies (a). Assume that f is not g-integrable on M or $f|M$ and $g|M$ have a common point of discontinuity.

First, assume that f is not g-integrable on M. By Theorem 1, there is a positive number ε such that for each partition D of M, there is a refinement D' of D, a choice function δ_D on D, and a choice function $\delta_{D'}$ on D' such that

$$|\Sigma(f, g, D, \delta_D) - \Sigma(f, g, D', \delta_{D'})| \geqslant \varepsilon .$$

Using the preceding information, it can be shown that there is a sequence $D_1, D_2, \ldots$ of partitions of M and a sequence of refinements $D_1', D_2', D_3', \ldots$ of $D_1, D_2, D_3, \ldots$, respectively, satisfying the following conditions:

1) $\big(E(D_n) \cup E(D_n') \cup \delta_{D_n}(D_n) \cup \delta_{D_n'}(D_n')\big) \subset E(D_{n+1}), \qquad n = 1, 2, 3, \ldots .$

2) $\bigcup_{i=1}^{\infty} E(D_i) \in \Delta$.

3) $|\Sigma(f, g, D_n, \delta_{D_n}) - \Sigma(f, g, D_n', \delta_{D_n'})| \geqslant \varepsilon, \; n = 1, 2, 3, \ldots :$

$\bigcup_{i=1}^{\infty} E(D_i)$ is a countable member of Δ and a subset of M, so f is g-integrable on $\bigcup_{i=1}^{\infty} E(D_i)$. There is a partition F of $\bigcup_{i=1}^{\infty} E(D_i)$ such that $|\int_{\bigcup_{i=1}^{\infty} E(D_i)} f dg - \sum(f, g, F', \delta)| < \varepsilon/2$ for each refinement F' of F and for each choice function δ on F'.

Since F is finite and each member of F is a member of $E(D_n)$ for some n, there is a positive integer N such that $E(F) \subset E(D_p)$ for each $p \geqslant N$. Since $E(F) \subset E(D_N)$, let F_N be a refinement of F such that $E(F_N) = E(D_N)$, and let F_N' be a refinement of F such that $E(F_N') = E(D_N')$. If $d \in F_N$, let d'

denote a member of D_N such that d and d' have the same end points. Let δ_{F_N} be a function with domain F_N given by $\delta_{F_N}(d) = \delta_{D_N}(d')$ for each d in F_N. Now, let d be an arbitrary member of F_N. $\delta_{D_N}(d') \in d'$ and $\delta_{D_N}(d') \in \bigcup_{i=1}^{\infty} E(D_i)$.

Thus, $\delta_{F_N}(d) = \delta_{D_N}(d') \in d$. δ_{F_N} is a choice function on F_N. Define $\delta_{F_N'}$ in like manner so that $\delta_{F_N'}$ will be a choice function on F_N. It should now be reasonably clear that $\Sigma(f, g, F_N, \delta_{F_N}) = \Sigma(f, g, D_N, \delta_{D_N})$ and $\Sigma(f, g, F_N', \delta_{F_N'}) = \Sigma(f, g, D_N', \delta_{D_N'})$. We have

$$\Big| \int_{\bigcup_{i=1}^{\infty} E(D_i)} f\, dg - \Sigma(f, g, D_N, \delta_{D_N}) \Big| < \varepsilon/2$$

and

$$\Big| \int_{\bigcup_{i=1}^{\infty} E(D_i)} f\, dg - \Sigma(f, g, D_N', \delta_{D_N'}) \Big| < \varepsilon/2 .$$

The preceding yields

$$|\Sigma(f, g, D_N, \delta_{D_N}) - \Sigma(f, g, D_N', \delta_{D_N'})| < \varepsilon$$

which contradicts condition 3) above.

Therefore f is g-integrable on M.

Now, assume that $f|M$ and $g|M$ have a common point of discontinuity which we will call z. This allows one to determine a countable member $M' \in \Delta$ and $k > 0$ such that for each open interval s containing z, $|f(x) - f(y)| \cdot |g(q) - g(p)| > k$ for some p and some q belonging to $S \cap M'$ and for some x and some y between p and q where x and y belong to M'. A simple application of the definition of integral shows that this is enough to make f not g-integrable on M', which is a contradiction.

Thus (b) implies (a).

Now, suppose M' is a countable subset of M and $M' \in \Delta$.

Let $\varepsilon > 0$. There is a partition D of M such that

$$\Big| \int_M f dg - \sum (f, g, D', \delta) \Big| < \varepsilon/2 \tag{1}$$

for each refinement D' of D and each choice function δ on D'.

It will be sufficient to consider the case $E(D) \not\subset M'$. Then $E(D) \setminus M' = \{z_1, z_2, \ldots, z_N\}$ is a nonempty subset of $M \setminus M'$. Using the fact that at each element of $E(D)\setminus M'$ either $f|M$ or $g|M$ is continuous and the fact that f and g are bounded on M, it can be shown that there is $d > 0$ such that, if $z \in E(D)\setminus M'$, then

$$|f(u) - f(v)| \cdot |g(s) - g(r)| < \varepsilon/2N \tag{2}$$

for each r, s, u, and v in $(z - d, z + d) \cap M$ such that $r < z < s$ and $r \leqslant u < v \leqslant s$.

Let D_1 denote a refinement of D such that $\big(E(D_1)\setminus E(D)\big) \subset M'$, $\|D_1\| < d/2$ $\big(\|D_1\| = \sup\{x \mid \text{there is a } w \text{ in } D_1 \text{ and } p,\ q \in w \text{ such that } p < q \text{ and } x = q - p\}\big)$ and between such two points of $E(D)\setminus M'$ there is a point of $E(D_1)$. Also, let F be a partition of M' each that $E(F) = \big(E(D_1)\setminus M\big) \cup \{a, b\}$.

Suppose that F' is a refinement of F and δ is a choice function on F'. A few preliminaries are in order:

D_1' denotes a refinement of D_1 such that $E(D_1') = E(F') \cup E(D_1)$.
G_1 is a subset of D_1' such that

$$G_1 = \big\{[p_1, z_1] \cap M, [p_2, z_2] \cap M, \ldots, [p_N, z_N] \cap M\big\}.$$

G_2 is a subset of D_1' such that

$$G_2 = \big\{[z_1, q_1] \cap M, [z_2, q_2] \cap M, \ldots, [z_N, q_N] \cap M\big\}.$$

δ' is a choice function on D_1' such that if $d' \in D_1'\setminus(G_1 \cup G_2)$, then $\delta'(d') = \delta(d)$ where d is a member of F' with the same end points as d', and if $d' \in G_1 \cup G_2$, then $\delta'(d')$ is the end point of d' which is in $E(D)\setminus M'$.

From (1), we have

$$\Big|\int_M f\,dg - \sum(f, g, D_1'\setminus(G_1 \cup G_2), \delta') - \sum(f, g, G_1, \delta') - \sum(f, g, G_2, \delta')\Big| < \varepsilon/2, \tag{3}$$

which yields

$$\Big|\int_M f\,dg - \sum(f, g, F'\setminus G', \delta) - \sum(f, g, G_1, \delta') - \sum(f, g, G_2, \delta')\Big| < \varepsilon/2, \tag{4}$$

where G' is the set of all members of F' which have a point of $E(D)\setminus M'$ between their end points.

By (2)

$$(5)\quad \begin{cases} |\Sigma(f, g, G_1, \delta') + \Sigma(f, g, G_2, \delta') - \sum(f, g, G', \delta)| = \\ = |\sum_{i=1}^{N} [f(z_i)\cdot(g(z_i) - g(p_i)) + f(z_i)\cdot(g(q_i) - q(z_i))] - \\ \qquad - \sum_{i=1}^{N} f(\delta([p_i, q_i] \cap M'))\cdot(g(q_i) - g(p_i))| \leqslant \\ \leqslant \sum_{i=1}^{N} |f(z_i) - f(\delta([p_i, q_i] \cap M'))| \cdot |g(q_i) - g(p_i)| < \varepsilon/2\,. \end{cases}$$

Combining (4) and (5), we have

$$|\int_{M} f\,\mathrm{d}g - \sum (f, g, F', \delta)| < \varepsilon\,.$$

So f is g-integrable on M', and (a) implies (b).

References.

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[4] J. F. Vance, *A representation theorem for bounded linear functionals*, Ph. D. thesis, Univ. of Texas Library, Austin, Texas, 1967.

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