Riv. Mat. Univ. Parma (3) 1 (1972), 41-46

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Integrals Involving Bessel Polynomials. (**)

Introduction.

Little is known about integrals involving BESSEL polynomials which arise as a solution of the classical wave equation in spherical coordinates. These polynomials are defined by KRALL and FRINK [1] by

$$y_n(x, a, b) = {}_2F_0(-n, a+n-1; -x/b)\,. \tag{1}$$

All is known are series expansions of these polynomials. These expansions were given by several writers (BRAFMAN [2], RAGAB [3], AL-SALAM [4], [5] and AGARWAL [6]). The purpose of this paper is to present some integrals involving these polynomials. These integrals will be stated in section (2) while their proofs will be given in (3). Integrals showing the orthogonal properties of these polynomials will be stated in (4).

1. - Formulae required in the proofs.

GAUSS's theorem ([7], p. 28), namely:

$$F\begin{pmatrix}\alpha, \beta;\ 1\\ \gamma\end{pmatrix} = \frac{\Gamma(\gamma)\,\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)}\,, \tag{2}$$

where $\mathrm{Re}\,\gamma > 0$, $\mathrm{Re}\,(\gamma-\alpha-\beta) > 0$.

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[8] (p. 249), namely:

$$(3)\quad \begin{cases} F\begin{pmatrix}\alpha,\beta;\,x\\ \gamma\end{pmatrix} = \dfrac{\Gamma(\gamma-\alpha-\beta)\,\Gamma(\gamma)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)}\,F\begin{pmatrix}\alpha,\beta;\,1-x\\ \alpha+\beta-\gamma+1\end{pmatrix} + \\ +\dfrac{\Gamma(\alpha+\beta-\gamma)\,\Gamma(\gamma)}{\Gamma(\alpha)\,\Gamma(\beta)}\,(1-x)^{\gamma-\alpha-\beta}\,F\begin{pmatrix}\gamma-\alpha,\,\gamma-\beta;\,1-x\\ \gamma-\alpha-\beta+1\end{pmatrix}. \end{cases}$$

Frequent use will be made of the factorial notation

$$(\alpha;n) = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} = \alpha(\alpha+1)(\alpha+2)\ldots(\alpha+n-1) \qquad (n=0,1,2,\ldots),$$

with $(\alpha;0)=1$. The relation

$$(4)\qquad (\alpha;-n) = (-1)^n/(1-\alpha;n)$$

will also be utilised.

[8] (p. 397, ex. 63), namely:

$$(5)\qquad \int_0^\infty K_n(\lambda)\,\lambda^{m-1}\,d\lambda = 2^{m-1}\Gamma\left(\frac{m+n}{2}\right)\Gamma\left(\frac{m-n}{2}\right),$$

where $\operatorname{Re}(m\pm n)>0$ and $K_n(\lambda)$ is the modified BESSEL function of the second kind.

[8] (p. 397, ex. 61), namely:

$$(6)\qquad \int_0^\infty J_n(\lambda)\,\lambda^{m-1}\,d\lambda = 2^{m-1}\left\{\Gamma\left(\frac{m+n}{2}\right)\Big/\Gamma\left(1+\frac{n-m}{2}\right)\right\},$$

where $\operatorname{Re}(m+n)>0$ and $\operatorname{Re}(m)<3/2$ and $J_n(\lambda)$ is the BESSEL function of the first kind.

[10], namely:

$$(7)\qquad \int_0^\infty e^{-t}t^{\gamma-1}y_n(y,a,t)\,y_m(x,b,t)\,dt = \frac{\Gamma(\gamma-n)\,\Gamma((a+\gamma+n-1)}{\Gamma(a+\gamma-1)},$$

where $\operatorname{Re}(\gamma-n)>0$, $\operatorname{Re}(a+\gamma+n-1)>0$.

2. - Integrals involving Bessel polynomials.

The formulae to be proved are:

$$\int_0^1 x^{1-\alpha-n}(1-x)^{\alpha-\beta-1}\, y_n(y,\alpha,bx)\,\mathrm{d}x = \frac{\Gamma(2-\alpha-n)\,\Gamma(\alpha-\beta)}{\Gamma(2-\beta-n)}\, y_n(y,\beta,b)\,, \tag{8}$$

where $\mathrm{Re}\,(2-\alpha-2n)>0$, $\mathrm{Re}\,(2-\beta-2n)>0$, $\mathrm{Re}\,(\alpha-\beta)>0$;

$$\left\{\begin{aligned} &\int_0^1 e^{-t}t^{k-1}y_n(1,a,t)\,{}_pF_q\binom{\alpha_1,\ldots,\alpha_p;\ \lambda t}{\varepsilon_1,\ldots,\varepsilon_q}\,\mathrm{d}t = \\ &= \Gamma(k-n)\cdot(k+a-1;\,n)\;{}_{p+2}F_{q+1}\binom{\alpha_1,\ldots,\alpha_p;\ k-n,\,a+n+k-1;\ \lambda}{\varepsilon_1,\ldots,\varepsilon_q;\ a+k-1}\,, \end{aligned}\right. \tag{9}$$

where $\mathrm{Re}\,(k-n)>0$ and $p\leqslant q+1$;

$$\left\{\begin{aligned} &\int_0^\infty e^{-t}t^{\gamma-1}y_n(1,a,t)\,{}_pF_q\binom{\alpha_1,\ldots,\alpha_p;\ \lambda/t}{\varepsilon_1,\ldots,\varepsilon_q}\,\mathrm{d}t = \\ &= \Gamma(\gamma-n)(\gamma+a-1;\,n)\;{}_{p+1}F_{q+2}\binom{\alpha_1,\ldots,\alpha_p,\,2-\alpha-\gamma;\ -\lambda}{\varepsilon_1,\ldots,\varepsilon_q,\,2-a-\gamma-n,\,1+n-\gamma}\,, \end{aligned}\right. \tag{10}$$

where $\mathrm{Re}\,(\gamma-2n)>0$ and $p\leqslant q+1$;

$$\left\{\begin{aligned} &\int_0^\infty \lambda^{\gamma-1}J_\mu(\lambda)\,y_n(b,a,\lambda^2)\,\mathrm{d}\lambda = \\ &\qquad = 2^{\gamma-1}\frac{\Gamma\left(\dfrac{\gamma+\mu}{2}\right)}{\Gamma\left(1+\dfrac{\mu-\gamma}{2}\right)}\;{}_2F_2\left(\begin{matrix}-n,\ a+n-1;\ b/4\\ 1-\dfrac{\gamma+\mu}{2},\ 1+\dfrac{\mu-\gamma}{2}\end{matrix}\right), \end{aligned}\right. \tag{11}$$

where $\mathrm{Re}\,(\gamma+\mu)>2n$; $\mathrm{Re}\,(\gamma)<\frac{1}{2}$.

$$\left\{\begin{aligned} &\int_0^\infty x^{\gamma-1}J_\mu(x)\,y_n(b,a;x)\,y_n(-b,a;x)\,\mathrm{d}x = \\ &\qquad = \frac{2^{\gamma-1}\Gamma\left(\dfrac{\gamma+\mu}{2}\right)}{\Gamma\left(1-\dfrac{\mu-\gamma}{2}\right)}\;{}_4F_3\left(\begin{matrix}-n,\ a+n-1,\dfrac{a}{2}-\dfrac{1}{2},\dfrac{a}{2};\ -b^2\\ a-1,\ 1-\dfrac{\gamma+\mu}{2},\ 1+\dfrac{\mu-\gamma}{2}\end{matrix}\right), \end{aligned}\right. \tag{12}$$

where $\operatorname{Re}(\gamma+\mu)>0$; $\operatorname{Re}(\gamma)<\frac{1}{2}$;

$$(13)\quad \begin{cases} \int\limits_0^\infty e^{-t}\, t^{\gamma-1} y_n(1, a, t)\, y_m(t, \alpha, b)\, y_m(-t, \alpha, b)\, \mathrm{d}t = \\ \qquad = 2^{\gamma-1}\Gamma(\gamma-n)\cdot(\gamma+a-1; n)\cdot \\ \cdot\, {}_8F_3\left(\begin{matrix} -m,\ \alpha+m-1,\ \frac{\alpha-1}{2},\ \frac{\alpha}{2},\ \frac{\gamma-n}{2},\ \frac{\gamma-n+1}{2},\ \frac{\gamma+a-1+n}{2},\ \frac{\gamma+a+n}{2};\ \frac{16}{b^2} \\ \alpha-1,\ \frac{\gamma+a-1}{2},\ \frac{\gamma+a}{2} \end{matrix}\right), \end{cases}$$

where $\operatorname{Re}(\gamma-n+2m)>0$;

$$(14)\quad \begin{cases} \int\limits_0^\infty e^{-t}\, t^{\gamma-1} y_n(1, a, t)\, y_m(b, \alpha, t)\, y_m(-b, \alpha, t)\, \mathrm{d}t = \\ \qquad = \Gamma(\gamma-n)(a+\gamma-1; n)\cdot \\ \cdot\, {}_6F_5\left(\begin{matrix} -m,\ \alpha+m-1,\ \frac{\alpha}{2}-\frac{1}{2},\ \frac{\alpha}{2},\ 1-\frac{\gamma+a-1}{2},\ 1-\frac{\gamma+a}{2};\ b^2 \\ \alpha-1,\ 1-\frac{\gamma-n}{2},\ \frac{1}{2}-\frac{\gamma-n}{2},\ 1-\frac{\gamma+n+a-1}{2},\ 1-\frac{\gamma+a+n}{2} \end{matrix}\right), \end{cases}$$

where $\operatorname{Re}(\gamma-n-2m)>0$.

3. - Proofs of the formulae.

To prove (8), expand y_n by means of (1); change the order of integration and summation, then the left hand side becomes:

$$\sum_{r=0}^{n} \frac{(-n; r)(\alpha+n-1; r)}{r!}\left(\frac{-y}{b}\right)^r \int_0^1 x^{1-\alpha-n-r}(1-x)^{\alpha-\beta-1}\,\mathrm{d}x =$$

$$= \sum_{r=0}^{n} \frac{(n; r)(\alpha+n-1; r)}{r!}\left(\frac{-y}{b}\right)^r B(2-\alpha-n-r, \alpha-\beta).$$

The result now follows from (4). Thus (13) is proved.

To prove (9), expand the generalized hypergeometric function, change the order of integration and summation; then the left hand side of (9) becomes

$$\sum_{r=0}^{\infty} \frac{(\alpha_1; r) \dots (\alpha_p; r)}{r (\varepsilon_1; r) \dots (\varepsilon_q; r)} \lambda^r \int_0^\infty e^{-t} t^{k+r-1} y_n(1, a, t)\, \mathrm{d}t .$$

Here evaluate the last integral by means of (7) and so obtain the right, hand side of (9).

In the same way (10) can be obtained by applying (4).

To prove (11), expand y_n by means of (1), change the order of integration and summation, the left hand side of (11) becomes:

$$\sum_{r=0}^{n} \frac{(-n; r)(a + n - 1; r)}{r!} (b)^r \int_0^\infty \lambda^{\gamma-2r-1} J_\mu(\lambda)\, \mathrm{d}\lambda .$$

Here apply (6) and (4) to obtain the right hand side of (11). Thus (11) is proved.

To prove (12): apply the formula (Bailey [9]), namely:

$$(15) \quad {}_2F_0(\alpha, \beta; x)\, {}_2F_0(\alpha, \beta; -x) = {}_4F_1 \begin{pmatrix} \alpha, \beta, \dfrac{\alpha+\beta}{2}, \dfrac{\alpha+\beta+1}{2}; 4x^2 \\ \alpha+\beta \end{pmatrix} ;$$

to obtain

$$(16) \quad y_n(x; a; b)\, y_n(-x, a, b) = {}_4F_1 \begin{pmatrix} -n, a+n-1, \dfrac{a}{2} - \dfrac{1}{2}, \dfrac{a}{2}; \dfrac{4x^2}{b^2} \\ a-1 \end{pmatrix} .$$

Change the order of integration and summation, then the left hand side of (12) becomes:

$$\sum_{r=0}^{n} \frac{(-n; r)(a + n - 1; r)\left(\dfrac{a}{2} - \dfrac{1}{2}; r\right)\left(\dfrac{a}{2}; r\right)}{r!} (4b^2)^r \int_0^\infty x^{\gamma-2r-1} J_\mu(x)\, \mathrm{d}x .$$

Now evaluate the last integral by means of (6); apply (4) and so obtain the right hand side of (12). Thus (12) is proved.

To prove (13): apply (16) again and change the order of integration and summation, then the left hand side becomes

$$\sum_{r=0}^{n} \frac{(-n; r)(\alpha + n - 1; r)\left(\dfrac{\alpha}{2} - \dfrac{1}{2}; r\right)\left(\dfrac{\alpha}{2}; r\right)}{r!} \left(\frac{4}{b^2}\right)^r \int_0^\infty e^{-t}\, t^{\gamma+2r-1} y_n(1, a, t)\, \mathrm{d}t .$$

Now evaluate the last integral by means of (7) and so obtain the right hand side of (13). This completes the proofs of (13).

In the same way (14) can be obtained by applying (7) and (4).

4. – For brievity I mention the following new formula for the orthogonality of the BESSEL polynomials:

$$\int_0^1 x^{1-a} e^{-x} y_n(1, a, x)\, y_m(1, a, x)\, dx = \begin{cases} 0 & \text{if } m \neq n \\ n!\,\Gamma(2-a-n) & \text{if } m = n, \end{cases} \tag{17}$$

where the weight function is $x^{1-a} e^{-x}$, $\mathrm{Re}\,(\gamma - n) > 0$.

It may be noted that the last result is not known in the literature.

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Abstract.

In this paper many definite integrals involving the Bessel polynomials $y_n(x, a, b)$ *are evaluated, where* $y_n(x, a, b) = {}_2F_0\left(-n, a+n-1; \dfrac{-x}{b}\right)$.

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