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Fixed Point Theorems in Metric Spaces. (**)

A mapping T of a metric space X into itself is said to satisfy LIPSCHITZ condition with LIPSCHITZ constant k if

$$d(Tx, Ty) \leqslant k\ d(x, y) \qquad \text{for all } x, y \text{ in } X.$$

In case $0 \leqslant k < 1$, then T is called a contraction mapping. A well-known theorem of BANACH states that if X is a complete metric space and T is a contraction mapping of X into itself, then T has a unique fixed point.

The assumption $d(Tx, Ty) < d(x, y)$ is not sufficient for the existence of a fixed point in a complete metric space. For example, let $X = \{x \mid x \geqslant 1\}$ with the usual distance

$$d(x, y) = |x - y| ,$$

and let T: $X \to X$ be defined by

$$Tx = x + (1/x) .$$

Then $d(Tx, Ty) < d(x, y)$, $x \neq y$, but T has no fixed point [**1**]. However, if the space is compact then there is always a fixed point for such a mapping [**4**].

In this paper we have proved a general result of BANACH contraction principle. The results given earlier by CHU and DIAZ [**2**], EDELSTEIN [**3**], RAKOTH [**7**] and K. L. SINGH [**8**] will be easy corollaries to our work. In the end, some results related to sequence of mappings and fixed points have been given.

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Definition 1. We denote by $\mathcal{F}$ the family of fuctions $\lambda(x, y)$ satisfying the following conditions:

(1) $\lambda(x, y) = \lambda(d(x, y))$, i.e. λ is dependent on the distance between x and y.

(2) $0 \leqslant \lambda(d) < 1$, for every $d > 0$.

(3) $\lambda(d)$ is monotonically decreasing function of d.

Definition 2. A finite sequence $x_0, x_1, x_2, \ldots, x_n$ of points of X is called an ε-chain joining x_0 and x_n if $d(x_{i-1}, x_i) < \varepsilon$ for each $\varepsilon > 0$ $(i = 1, 2, \ldots, n)$.

Definition 3. A metric space X is said to be ε-chainable (well-linked) if for each pair (x, y) of its points there exists an ε-chain joining x and y.

Every connected metric space is well-linked, but the converse is not always true. However, for compact spaces both are equivalent [**6**].

Theorem 1. *If T is a mapping of a complete ε-chainable metric space X into itself satisfying $d(x, y) < \varepsilon$ implies that $d(T^m x, T^m y) \leqslant \lambda(x, y)\, d(x, y)$ for every x, y in X, for positive integer m and $\lambda(x, y) \in \mathcal{F}$, then T has a unique fixed point.*

Proof. Since (X, d) is ε-chainable, we define for every x, y in X:

$$d_\varepsilon(x, y) = \inf \sum_{i=1}^{n} d(x_{i-1}, x_i) ,$$

where the infimum is taken over all ε-chains $x_0, x_1, \ldots, x_n$ joining $x = x_0$ and $y = x_n$. Then d_ε is a distance function satisfying

(i) $$d(x, y) \leqslant d_\varepsilon(x, y) ,$$

(ii) $$d(x, y) = d_\varepsilon(x, y) \qquad \text{for } d(x, y) < \varepsilon.$$

From (ii) it follows that a sequence $\{x_n\}$ in X is a Cauchy sequence with respect to d_ε if and only if it is a Cauchy sequence with respect to d and it is convergent with respect to d_ε if and only if it is convergent with respect to d. Hence, (X, d_ε) is a complete metric space, becouse (X, d) is a complete metric space [**1**]. Since T^m satisfies the condition

$$d_\varepsilon(T^m x, T^m y) \leqslant \lambda(x, y)\, d_\varepsilon(x, y) \qquad \text{for all } x,\ y \text{ in } X,$$

and therefore by a corollary given by Rakotch [**7**] we get T^m has a unique fixed point.

It follows easily that T has a unique fixed point.

Remarks:

(1) In case $m=1$, and X is ε-chainable complete metric space, then we get a known result due to SINGH [8].

(2) In case $m=1$, X is a complete metric space and

$$d(Tx, Ty) \leqslant \lambda(x, y)\, d(x, y)\,,$$

we get a well-known result due to RAKOTCH [7].

(3) In case $\lambda(x, y) = k$, where $0 \leqslant k < 1$ and $m = 1$, then we get a result due to EDELSTEIN [3].

(4) In case $\lambda(x, y) = k$, where $0 \leqslant k < 1$, and X is a complete metric space and $T^m\colon X \to X$, such that

$$d(T^m x, T^m y) \leqslant k\; d(x, y)\,,$$

then we get a well-known result due to CHU and DIAZ [2].

We prove the following theorem on sequence of commuting family of mappings and common fixed points.

Theorem 2. *Let (X, d) be a complete metric space and let T_i $(i = 1, 2, \ldots)$ be a sequence of mappings of X into itself satisfying the following conditions:*

(i) *There exist c and k $(c > 0,\ 0 \leqslant k < 1)$ such that*

$$d(T_i\, x,\ T_i\, y) \leqslant k\; d(x, y) \qquad (i = 1, 2, \ldots)$$

whenever $d(x, y) \leqslant c$,

(ii) $$T_i T_j = T_j T_i \qquad (i, j = 1, 2, \ldots)\,.$$

Then the family T_i $(i = 1, 2, \ldots)$ has a common fixed point.

In the proof of this Theorem we need a definition and a theorem due to EDELSTEIN [5].

Definition 4. A metric space (X, d) is called weakly ε-chainable if together with a, b in X, there exists a sequence $C(a, b) = (a = x_0, \ldots, x_k = b)$ in X such that

$$d(x_{i-1}, x_i) \leqslant \varepsilon \qquad (i = 1, 2, \ldots, k)\,.$$

Theorem (EDELSTEIN [5]). If $T\colon X \to X$ is a mapping of a complete, weakly ε-chainable metric space (X, d) satisfying the condition $d(a, b) \leqslant \varepsilon$ implies that $d(Ta, Tb) \leqslant K d(a, b)$ for a, b in X and $0 \leqslant K < 1$, then there exists a unique z in X such that $Tz = z$.

Proof of Theorem 2. Let Y denote the set of all y in X with the property that a sequence $C(y, x_n)$ in X exists where

$$C(y, x_n) = \{y = a_0, a_1, \ldots, a_m = x_n\}$$

with $d(a_{i-1}, a_i) \leqslant c$ $(i = 1, 2, \ldots, m)$. Then Y is a closed metric subspace of X and $T_i(Y) \subset Y$. Using the above result due to EDELSTEIN [5], we get that for each i, a unique p_i in Y exists such that $T_i p_i = p_i$. We need to prove that p_i is a common fixed point for the family $\{T_i\}$.

Since $T_i T_j = T_j T_i$ for i, $j = 1, 2, \ldots$, and p_i is a unique fixed point for T_i, it follows that p_i is a common fixed point for the family $\{T_i\}$ by the commuting property.

Thus the proof.

References.

[1] F. F. BONSALL, *On some fixed point theorems of functional analysis*, Tata Institute Fundam. Research, Bombay 1962.

[2] S. C. CHU and J. B. DIAZ, *Rermarks on a generalization of Banach's principle of contraction mappings*, J. Math. Anal. Appl. **11** (1965), 440-446.

[3] M. EDELSTEIN, *An extension of Banach's contraction principle*, Proc. Amer. Math. Soc. **12** (1961), 7-10.

[4] M. EDELSTEIN, *On fixed and periodic points under contractive mappings*, J. London Math. Soc. **37** (1962), 74-79.

[5] M. EDELSTEIN, *A remark on a theorem of A. F. Monna*, Indag. Math. **26** (1964), 88-89.

[6] S. B. NADLER (jr.), *Sequences of contractions and fixed points*, Pacif. J. Math. **27** (1968), 579-585.

[7] E. RAKOTCH, *A note on contractive mappings*, Proc. Amer. Math. Soc. **13** (1962), 459-465.

[8] K. L. SINGH, *Some fixed point theorems in analysis*, Master's Thesis, Memorial University (1968).

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