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On Reducing a Quartic. (**)

Theory of Diophantine equations can sometimes be used to discuss the reducibility or otherwise of a given polynomial having rational coefficients [1] (cfr. Chap. 7). However this method demands the constant term to be a specified form and is not generally applicable even then. One may ask the question, « Given an arbitrary polynomial $P(x)$ of degree $n \geqslant 2$, is it possible to determine in a finite number of steps whether or not $P(x)$ is reducible? » The answer to this question for $n = 2$ or 3 is trivial, for a cyclotomic polynomial is in the affirmative, it being always irreducible [1] (cfr. Chap. 5), we provide an answer for $n = 4$ and for higher n the answer is not known at present.

Any quartic

$$P(x) \equiv a_0 x^4 + a_1 x^3 + a_2 x^2 + a_3 x + a_4 \qquad (a\text{'s rational})$$

can always be transformed into

$$P'(x') \equiv k\,(x'^4 + Ax'^2 + Bx' + C) \qquad (k \text{ rational and } A,\ B \text{ and } C \text{ integers})$$

by a transformation

$$x' = K\,(x + (a_1/4a_0))$$

where K is a suitable integer. Designate

$$Q(x) \equiv x^4 + Ax^2 + Bx + C\,.$$

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It is obvious that to discuss the reducibility of $P(x)$ it is sufficient to consider only $Q(x)$. Evidently if $Q(x)$ is reducible it can have either a linear factor and a cubic one or two quadratic factors. The cases of two linear and one quadratic and all four linear factors are contained in the second possibility. It can be proved (cfr. [1], p. 161) that these factors will necessarily have integral coefficients. If $Q(x)$ has a linear factor it can always be found by trial; as for the other possibility we prove the following

Theorem. *A necessary and sufficient condition that $Q(x)$ be reducible to two quadratic factors is that there exists an integer l such that*

$$(1) \qquad \begin{cases} \text{(i)} & l \text{ is a divisor of } B \\ \text{(ii)} & (A + l^2)^2 - (B/l)^2 = 4C \,. \end{cases}$$

Proof. The condition is sufficient. Suppose there exists such an integer l, then it is easy to verify that

$$Q(x) \equiv \left\{x^2 + lx + \frac{1}{2}\left(A + l^2 - \frac{B}{l}\right)\right\}\left\{x^2 - lx + \frac{1}{2}\left(A + l^2 + \frac{B}{l}\right)\right\}.$$

To prove that the condition is necessary as well, suppose

$$Q(x) \equiv (x^2 + px + q)(x^2 - px + r) \qquad (p, q, r \text{ integers}).$$

Then we must have

$$\text{(a)} \qquad qr = C\,,$$

$$\text{(b)} \qquad -p^2 + q + r = A\,,$$

$$\text{(c)} \qquad p(r - q) = B\,.$$

Eliminating q and r from (a), (b) and (c) we obtain

$$(A + p^2)^2 - (B/p)^2 = 4C\,.$$

Also (c) tells us that p is a divisor of B. Hence if $Q(x)$ be reducible to quadratic factors then there must be an integer which satisfies conditions (1).

Corollary 1. *$Q(x)$ cannot be resolved into quadratic factors if any one of the following conditions holds:*

1) *Both A and B are odd.*
2) *Both A and B are even but B is not divisible by 4.*
3) *$A \geqslant 0$ and $A^2 - B^2 \geqslant 4C$.*

We can now prove the following concerning the cubic

$$C(x) \equiv x^3 + Dx^2 + Ex + F .$$

Corollary 2. *$C(x)$ is irreducible if any one of the following is true:*

1) *F is odd and one of D and E is odd while the other is even.*
2) *D, E, F are all odd or are all even and $F - ED$ is not divisible by 4.*

Proof. Multiply $C(x)$ by $(x - D)$. We get

$$(x - D)\,C(x) \equiv x^4 + (E - D^2)\,x^2 + (F - ED)\,x - DF . \tag{2}$$

Now if $C(x)$ is reducible then the right hand side of (2) must be expressible into quadratic factors which, by Corollary 1, is impossible under the given conditions.

References.

[1] T. Nagell, *Introduction to Number Theory*, John Wiley & Sons, New York 1951.

Summary.

An elementary method of determining whether or not a given polynomial of the fourth degree (i.e. a quartic) is reducible is described.

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