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An Infinite Integral Involving Meijer's G-Function. (**)

1. – The object of the present paper is to evaluate an integral involving MEIJER's G-function with the help of a property of the LAPLACE transform established in § 2.

Throughout this note the conventional notation

$$\varphi(p) \doteqdot f(t)$$

will be used to denote the LAPLACE's integral

$$(1) \qquad \varphi(p) = p \int_0^\infty \exp(-pt) f(t)\, dt ,$$

provided that $\operatorname{Re} p > 0$ and the integral is convergent.

In what follows n is a positive integer and the symbol $\Delta(n;\alpha)$ will represent the sequence of parameters α/n, $(\alpha+1)/n$, $(\alpha+2)/n$, ..., $(\alpha+n-1)/n$.

2. - Theorem.

If

$$\varphi(p) \doteqdot f(t)$$

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and

$$\psi(p, \nu, \alpha, \beta) \doteqdot K_\nu(\alpha t)\, K_\nu(\beta t)\, f(t)\ ,$$

then

$$(2)\quad \begin{cases} \psi(p, \nu, \alpha, \beta) = \frac{1}{2}\pi\, p(\alpha\beta)^{1/2} \int\limits_1^\infty (\alpha^2 + \beta^2 + 2\alpha\beta t)^{-1/2} \cdot \\ \cdot\, [p + (\alpha^2 + \beta^2 + 2\alpha\beta t)^{1/2}]^{-1} P_{\nu-1/2}(t)\ \varphi[p + (\alpha^2 + \beta^2 + 2\alpha\beta t)^{1/2}]\, dt\ , \end{cases}$$

provided the integrals are absolutely convergent, $\mathrm{Re}\,\alpha > 0$, $\mathrm{Re}\,\beta > 0$, $\mathrm{Re}\,p > 0$ *and* $f(t)$ *is independent of* α *and* β.

Proof. From (1) it follows that

$$\int\limits_0^\infty \frac{\{(\alpha + \beta)^2 + 2\alpha\beta u\}^{-1/2} P_{\nu-1/2}(1 + u)}{p + \{(\alpha + \beta)^2 + 2\alpha\beta u\}^{1/2}}\ \varphi[p + \{(\alpha + \beta)^2 + 2\alpha\beta u\}^{1/2}]\, du =$$

$$= \int\limits_0^\infty P_{\nu-1/2}(1 + u)\, \{(\alpha+\beta)^2 + 2\alpha\beta u\}^{-1/2} \int\limits_0^\infty \exp(-px - x\,\{(\alpha+\beta)^2 + 2\alpha\beta u\}^{1/2}) \cdot f(x)\, dx\, du =$$

$$= \int\limits_0^\infty x \exp(-px)\, f(x) \int\limits_0^\infty P_{\nu-1/2}(1 + u)\, \{(\alpha x + \beta x)^2 + 2\alpha\beta u\, x^2\}^{-1/2} \cdot$$

$$\cdot \exp\,[-\{(\alpha x + \beta x)^2 + 2\alpha\beta u x^2\}^{1/2}]\, du\, dx.$$

On evaluating the u-integral by means of the formula ([4] p. 323)

$$(3)\quad \begin{cases} \int\limits_0^\infty \{(\alpha + \beta)^2 + 2\alpha\beta t\}^{-1/2} \exp\,[-\{(\alpha + \beta)^2 + 2\alpha\beta t\}^{1/2}]\, P_{\nu-1/2}(1 + t)\, dt = \\ = 2\pi^{-1}(\alpha\beta)^{-1/2} K_\nu(\alpha)\, K_\nu(\beta)\ , \end{cases}$$

where $\mathrm{Re}\,\alpha > 0$, $\mathrm{Re}\,\beta > 0$ and making the substitution $1 + u = t$ we arrive at the result.

The change of the order of integration in the above process is justified by virtue of a well-known theorem concerning the inversion of a repeated infinite integral ([1], p. 504).

On taking $\nu = \frac{1}{2}$ and using the identity

$$K_{\pm 1/2}(x) = (\pi/2x)^{1/2} e^{-x}\ ,$$

the theorem yields the following corollary.

Corollary. *If*

$$\varphi(p) \doteqdot f(t)\,,$$

and

$$\psi(p, \alpha, \beta) \doteqdot \frac{\pi}{2t}(\alpha\beta)^{-\frac{1}{2}} \exp(-(\alpha+\beta)t)\, f(t)\,,$$

then

$$(4)\quad \left\{ \begin{aligned} &\psi(p,\alpha,\beta) = \frac{\pi p}{2}(\alpha\beta)^{\frac{1}{2}} \int_1^\infty (\alpha^2+\beta^2+2\alpha\beta t)^{-\frac{1}{2}} \cdot \\ &\cdot [p + (\alpha^2+\beta^2+2\alpha\beta t)^{\frac{1}{2}}]^{-1}\, \varphi[p+(\alpha^2+\beta^2+2\alpha\beta t)^{\frac{1}{2}}]\,\mathrm{d}t\,, \end{aligned} \right.$$

provided the integrals are absolutely convergent, $\operatorname{Re}\alpha > 0$, $\operatorname{Re}\beta > 0$, $\operatorname{Re} p > 0$ *and* $f(t)$ *is independent of* α *and* β.

A result of a similar nature was obtained by Saxena ([**6**], p. 155).

3. - Example.

Starting from Saxena's formula ([**5**], p. 402, eq. 11) we have

$$f(t) = t^{-\lambda} G^{l,k}_{s,r}\left(at^n \,\middle|\, \begin{matrix} a_1, \dots, a_s \\ b_1, \dots, b_r \end{matrix}\right) \doteqdot p^\lambda n^{\frac{1}{2}-\lambda}\ (2\pi)^{\frac{1}{2}-\frac{1}{2}n} \cdot G^{l,k+n}_{s+n,r}\left[\frac{an^n}{p^n} \,\middle|\, \begin{matrix} \Delta(n,\lambda), a_1, \dots, a_s \\ b_1, \dots, b_r \end{matrix}\right],$$

where $\operatorname{Re}(nb_h - \lambda) > -1$ for $h = 1, 2, \dots, l$; $2(l+k) > r+s$, $0 \leqslant k \leqslant s$, $0 \leqslant l \leqslant r$, $|\arg a| < (k + l - \frac{1}{2}r - \frac{1}{2}s)\pi$, $\operatorname{Re} p > 0$.

We also have

$$\frac{\pi}{2}(\alpha\beta)^{-\frac{1}{2}}\, t^{-1} \exp(-(\alpha+\beta)t)\, f(t) =$$

$$= \frac{\pi}{2}(\alpha\beta)^{-\frac{1}{2}}\, t^{-\lambda-1} \exp(-(\alpha+\beta)t)\, G^{l,k}_{s,r}\left(at^n \,\middle|\, \begin{matrix} a_1, \dots, a_s \\ b_1, \dots, b_r \end{matrix}\right) =$$

$$= \frac{\pi}{2}(\alpha\beta)^{-\frac{1}{2}} p(p+\alpha+\beta)^\lambda n^{-\lambda-\frac{1}{2}} (2\pi)^{\frac{1}{2}-\frac{1}{2}n} \cdot$$

$$\cdot\, G^{l,k+n}_{s+n,r}\left[\frac{an^n}{(p+\alpha+\beta)^n} \,\middle|\, \begin{matrix} \Delta(n;\lambda+1), a_1, \dots, a_s \\ b_1, \dots, b_r \end{matrix}\right],$$

where $\operatorname{Re}(nb_h - \lambda) > 0$ for $h = 1, 2, \dots, l$, $\operatorname{Re}(p+\alpha+\beta) > 0$, $0 \leqslant k \leqslant s$, $0 \leqslant l \leqslant r$, $|\arg a| < (k + l - \frac{1}{2}r - \frac{1}{2}s)\pi$, $k + l > \frac{1}{2}r + \frac{1}{2}s$.

Applying (4) and making suitable changes in the parameters, it if found that

$$(5)\quad \begin{cases} \int_1^\infty (\alpha^2+\beta^2+2\alpha\beta t)^{-1/2}\,[\gamma+(\alpha^2+\beta^2+2\alpha\beta t)^{1/2}]^{\lambda-1}\,\cdot \\ \qquad \cdot\, G_{r,\,s}^{k,\,l}\left[\{\gamma+(\alpha^2+\beta^2+2\alpha\beta t)^{1/2}\}^n \,\middle|\, \begin{matrix} a_1,\dots,a_r \\ b_1,\dots,b_s \end{matrix}\right] dt = \\ = (n\alpha\beta)^{-1}(\alpha+\beta+\gamma)^\lambda\, G_{r+1,\,s+1}^{k+1,\,l}\left[(\alpha+\beta+\gamma)^n \,\middle|\, \begin{matrix} a_1,\dots,a_r,\ 1-\lambda/n \\ -\lambda/n,\ b_1,\dots,b_s \end{matrix}\right], \end{cases}$$

valid by analytic continuation for $\mathrm{Re}[\lambda+n(a_j-1)]<0$ for $j=1,2,\dots,l$, $2(k+l)>r+s$, $0\leqslant l\leqslant r$, $0\leqslant k\leqslant s$, $\mathrm{Re}\,\alpha>0$, $\mathrm{Re}\,\beta>0$, $\mathrm{Re}\,\gamma\geqslant 0$.

4. - Particular cases.

(i) when $\gamma=0$ (5) reduces to

$$(6)\quad \begin{cases} \int_1^\infty (\alpha^2+\beta^2+2\alpha\beta t)^{(\lambda/2)-1}\, G_{r,\,s}^{k,\,l}\left[(\alpha^2+\beta^2+2\alpha\beta t)^{n/2} \,\middle|\, \begin{matrix} a_1,\dots,a_r \\ b_1,\dots,b_s \end{matrix}\right] dt = \\ = (n\alpha\beta)^{-1}(\alpha+\beta)^\lambda\, G_{r+1,\,s+1}^{k+1,\,l}\left[(\alpha+\beta)^n \,\middle|\, \begin{matrix} a_1,\dots,a_r,\ 1-\lambda/n \\ -\lambda/n,\ b_1,\dots,b_s \end{matrix}\right], \end{cases}$$

where $\mathrm{Re}(\lambda+n\,a_j)<n-1$ for $j=1,2,\dots,l$, $\mathrm{Re}\,\alpha>0$, $\mathrm{Re}\,\beta>0$.

On taking $\alpha=\beta=1$, and $n=2$ in (6) it gives a particular case of the formula ([**3**], p. 338).

(ii) setting $k=s=2$, $l=r=0$, $b_1=\nu/2$ and $b_2=-\nu/2$ in (5) and applying ([**2**], p. 216, eq. 4) we find that

$$(7)\quad \begin{cases} \int_1^\infty (\alpha^2+\beta^2+2\alpha\beta t)^{-1/2}\,[\gamma+(\alpha^2+\beta^2+2\alpha\beta t)^{1/2}]^{\lambda-1}\,\cdot \\ \qquad \cdot\, K_\nu[2\{\gamma+(\alpha^2+\beta^2+2\alpha\beta t)^{1/2}\}^{n/2}]\,dt = \\ = \dfrac{(\alpha+\beta+\gamma)^\lambda}{2n\alpha\beta}\, G_{1,3}^{3,0}\left[(\alpha+\beta+\gamma)^n \,\middle|\, \begin{matrix} 1-\lambda/n \\ -\lambda/n,\ \nu/2,\ -\nu/2 \end{matrix}\right], \end{cases}$$

where $\mathrm{Re}\,\alpha>0$, $\mathrm{Re}\,\beta>0$ and $\mathrm{Re}\,\gamma\geqslant 0$.

If we put $\lambda = n(1-\nu/2)$ the above equation then reduces to

$$(8)\quad \left\{ \begin{aligned} & \int_1^\infty (\alpha^2+\beta^2+2\alpha\beta t)^{-1/2}[\gamma+(\alpha^2+\beta^2+2\alpha\beta t)^{1/2}]^{n(1-\nu/2)-1}\cdot \\ & \qquad \cdot K_\nu[2\{\gamma+(\alpha^2+\beta^2+2\alpha\beta t)^{1/2}\}^{n/2}]\,dt = \\ & = \frac{(\alpha+\beta+\gamma)^{1/2 n(1-\nu)}}{n\alpha\beta} K_{\nu-1}[2(\alpha+\beta+\gamma)^{n/2}], \end{aligned} \right.$$

where $\operatorname{Re}\alpha > 0$, $\operatorname{Re}\beta > 0$ and $\operatorname{Re} k \geqslant 0$.

(iii) Putting $k=1$, $l=r=0$, $s=2$, $b_1=\nu/2$ and $b_2=-\nu/2$ in (5) and applying ([**2**], p. 216, eq. 3) we see that

$$(9)\quad \left\{ \begin{aligned} & \int_1^\infty (\alpha^2+\beta^2+2\alpha\beta t)^{-1/2}[\gamma+(\alpha^2+\beta^2+2\alpha\beta t)^{1/2}]^{\lambda-1}\cdot \\ & \qquad \cdot J_\nu[2\{\gamma+(\alpha^2+\beta^2+2\alpha\beta t)^{1/2}\}^{n/2}]\,dt = \\ & = \frac{(\alpha+\beta+\gamma)^{\lambda}}{n\alpha\beta} G^{2,0}_{1,3}\left[(\alpha+\beta+\gamma)^n \,\middle|\, \begin{matrix} 1-\lambda/n \\ -\lambda/n,\ \nu/2,\ -\nu/2 \end{matrix}\right], \end{aligned} \right.$$

where α, $\beta > 0$ and $\operatorname{Re}\gamma \geqslant 0$.

If we set $\lambda = n(1-\nu/2)$ in (9) it yields

$$(10)\quad \left\{ \begin{aligned} & \int_1^\infty (\alpha^2+\beta^2+2\alpha\beta t)^{-1/2}[\gamma+(\alpha^2+\beta^2+2\alpha\beta t)^{1/2}]^{n(1-\nu/2)-1}\cdot \\ & \qquad \cdot J_\nu[2\{\gamma+(\alpha^2+\beta^2+2\alpha\beta t)^{1/2}\}^{n/2}]\,dt = \\ & = \frac{(\alpha+\beta+\gamma)^{(n/2)(1-\nu)}}{n\alpha\beta} J_{\nu-1}[2(\alpha+\beta+\gamma)^{n/2}], \end{aligned} \right.$$

where $\alpha, \beta > 0$ and $\operatorname{Re}\gamma \geqslant 0$.

(iv) Putting $k=s=2$, $l=0$, $r=1$, $a_1=1-k$, $b_1=\frac{1}{2}+m$, $b_2=\frac{1}{2}-m$ in (5) and applying ([**2**], p. 216, eq. 6) it is seen that

$$(11)\quad \left\{ \begin{aligned} & \int_1^\infty (\alpha^2+\beta^2+2\alpha\beta t)^{-1/2}[\gamma+(\alpha^2+\beta^2+2\alpha\beta t)^{1/2}]^{\lambda-1}\cdot \\ & \cdot \exp(-\tfrac{1}{2}[\gamma+(\alpha^2+\beta^2+2\alpha\beta t)^{1/2}]^n\, W_{k,m}[\{\gamma+(\alpha^2+\beta^2+2\alpha\beta t)^{1/2}\}^n]\,dt = \\ & = \frac{(\alpha+\beta+\gamma)^{\lambda}}{n\alpha\beta} G^{3,0}_{2,3}\left[(\alpha+\beta+\gamma)^n \,\middle|\, \begin{matrix} 1-k,\ 1-\lambda/n \\ -\lambda/n,\ \frac{1}{2}+m,\ \frac{1}{2}-m \end{matrix}\right], \end{aligned} \right.$$

where $\operatorname{Re}\alpha > 0$, $\operatorname{Re}\beta > 0$ and $\operatorname{Re}\gamma \geqslant 0$.

When $k = -m + \frac{1}{2}$ (11) reduces to

$$(12) \quad \begin{cases} \int_1^\infty (\alpha^2 + \beta^2 + 2\alpha\beta t)^{-\frac{1}{2}} [\gamma + (\alpha^2 + \beta^2 + 2\alpha\beta t)^{\frac{1}{2}}]^{\lambda-1} \cdot \\ \qquad \cdot \exp[-\{\gamma + (\alpha^2 + \beta^2 + 2\alpha\beta t)^{\frac{1}{2}}\}^n] \, dt = \\ \qquad = \Gamma\left[\frac{\lambda}{n}, (\alpha + \beta + \gamma)^n\right], \end{cases}$$

where $\operatorname{Re}\alpha > 0$, $\operatorname{Re}\beta > 0$, $\operatorname{Re}\gamma \geqslant 0$ and $\Gamma(\lambda, x)$ denotes an incomplete gamma function.

On the other hand, if we take $\lambda/n = \frac{1}{2} - m$ (11) reduces to

$$(13) \quad \begin{cases} \int_1^\infty (\alpha^2 + \beta^2 + 2\alpha\beta t)^{-\frac{1}{2}} [\gamma + (\alpha^2 + \beta^2 + 2\alpha\beta t)^{\frac{1}{2}}]^{n(\frac{1}{2}-m)-1} \cdot \\ \cdot \exp[-\frac{1}{2}\{\gamma + (\alpha^2 + \beta^2 + 2\alpha\beta t)^{\frac{1}{2}}\}^n] \, W_{k,m}[\{1 + (\alpha^2 + \beta^2 + 2\alpha\beta t)^{\frac{1}{2}}\}^n] \, dt = \\ = \frac{(\alpha + \beta + \gamma)^{-mn}}{n\alpha\beta} \exp(-\frac{1}{2}(\alpha + \beta + \gamma)^n) \, W_{k-\frac{1}{2}, m-\frac{1}{2}}[(\alpha + \beta + \gamma)^n], \end{cases}$$

where $\operatorname{Re}\alpha > 0$, $\operatorname{Re}\beta > 0$ and $\operatorname{Re}\gamma \geqslant 0$.

(v) Finally take $k = 1$, $l = r = s = 2$, $a_1 = 1 - a$, $a_2 = 1 - b$, $b_1 = 0$, $b_2 = 1 - c$ in (5) and apply ([**2**], p. 218, eq. 34) we then obtain

$$(14) \quad \begin{cases} \int_1^\infty (\alpha^2 + \beta^2 + 2\alpha\beta t)^{-\frac{1}{2}} [\gamma + (\alpha^2 + \beta^2 + 2\alpha\beta t)^{\frac{1}{2}}]^{\lambda-1} \cdot \\ \qquad \cdot {}_2F_1[a, b; c; -\{\gamma + (\alpha^2 + \beta^2 + 2\alpha\beta t)^{\frac{1}{2}}\}^n] \, dt = \\ \qquad = \frac{\Gamma(c)(\alpha + \beta + \gamma)^n}{n\alpha\beta\,\Gamma(a)\,\Gamma(b)} G^{2,2}_{3,3}\left[(\alpha + \beta + \gamma)^n \,\middle|\, \begin{matrix} 1 - a, \ 1 - b, \ 1 - \lambda/n \\ -\lambda/n, \ 0, \ 1 - c \end{matrix}\right], \end{cases}$$

where $\operatorname{Re}\alpha > 0$, $\operatorname{Re}\beta > 0$, $\operatorname{Re}\gamma \geqslant 0$, $\operatorname{Re}(\lambda - na) < 1$ and $\operatorname{Re}(\lambda - nb) < 1$.

References.

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