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Absolute Riesz Summability and a New Criterion for the Absolute Convergence of a Fourier Series. (**)

1. - Definitions and notations.

Let $\sum a_n$ be a given infinite series and $\{\lambda_n\}$ a positive; steadily increasing, monotonic sequence, tending to infinity with n.

The series $\sum a_n$ is said to be summable by RIESZ means of type λ_n and order r, or summable $(\mathrm{R}, \lambda_n, r)$, $r \geqslant 0$, to sum s (finite), if

$$R_\lambda^r(w) = w^{-r} \sum_{\lambda_n \leq w} (w - \lambda_n)^r a_n \to s \qquad \text{as} \quad w \to \infty \quad (^1) .$$

The series $\sum a_n$ is said to be absolutely summable $(\mathrm{R}, \lambda_n, r)$, or summable $|\mathrm{R}, \lambda_n\ r|$, $r > 0$, if

$$R_\lambda^r(w) \in \mathrm{BV}\,(h, \infty) \quad (^2) ,$$

where h is some finite positive number (3).

We suppose that $f(t)$ is a periodic function, with period 2π, integrable in the sense of LEBESGUE over $(-\pi, \pi)$. Without loss of generality, we assume

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(1) RIESZ [7].

(2) By « $f(x) \in \mathrm{BV}(h, k)$ » we mean that $f(x)$ is a function of bounded variation in (h, k).

(3) OBRECHKOFF [4], [5].

that the constant term in the FOURIER series of $f(t)$ is zero, so that

$$\int_{-\pi}^{\pi} f(t)\,dt = 0$$

and

$$f(t) \sim \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} A_n(t) .$$

Let

$$\varphi(t) = \tfrac{1}{2}\{f(x+t) + f(x-t)\} ,$$

so that

$$\varphi(t) \sim \sum_{n=1}^{\infty} A_n(x) \cos nt ,$$

and we study the absolute RIESZ summability and absolute convergence of $\sum_{n=1}^{\infty} A_n(x)$.

We follow the following notations throughout this paper. Let $0 < \alpha < 1$ and $\varepsilon > 0$.

(1.1) $$\beta(t) = \frac{1}{t} \int_0^t u \, d\varphi(u) ,$$

(1.2) $$\eta(w) = \sum_{\exp(n^\alpha) \leq w} \exp(n^\alpha)/n(\log(n+1))^p \qquad (p > 1)$$

(1.3) $$K(w, t) = \sum_{\exp(n^\alpha) \leq w} \exp(n^\alpha) \sin nt/n ,$$

(1.4) $$g(n, t) = \int_0^t u \left(\log \frac{k}{u}\right)^{-1-\varepsilon} \frac{\partial}{\partial u} \frac{\sin nu}{nu} \, du ,$$

(1.5) $$R(n, t) = \int_0^t u^{-1} \left(\log \frac{k}{u}\right)^{-p} \sin nu \, du \qquad (p > 1,\ 0 < t \leqslant \pi) .$$

2. - Introduction.

In 1950, MOHANTY (1) gave the following criterion for the absolute convergence of a LEBESGUE-FOURIER series at a point, which is the analogue for the absolute convergence of the classical HARDY-LITTLEWOOD convergence criterion (2).

Theorem A. If (i) $\varphi(t)\log(k/t) \in \mathrm{BV}(0,\pi)$, where $k \geqslant \pi e^2$, and (ii) the sequence $\{n^\delta A_n(x)\} \in \mathrm{BV}$, for $0<\delta<1$, then the series $\sum_{n=1}^{\infty} A_n(x)$ is absolutely convergent.

The technique used by MOHANTY was to obtain the following theorem on the absolute RIESZ summability of a FOURIER series at a point, and to deduce Theorem A by means of a Tauberian theorem, generalized later by PATI (3).

Theorem B. If $\varphi(t)\log(k/t) \in \mathrm{BV}(0,\pi)$, then the series $\sum_{n=1}^{\infty} A_n(x)$ is summable $|\mathrm{R}, \exp(n^\alpha), 1|$ $(0<\alpha<1)$.

In the present paper the author obtains absolute RIESZ summability of a FOURIER series of type $\exp(n^\alpha)$ $(0<\alpha<1)$, and order unity, and uses this result for obtaining a new criterion for the absolute convergence of a FOURIER series at a point. We establish the following theorems:

Theorem 1. *If* (i) $\varphi(t) \in \mathrm{BV}(0,\pi)$ *and* (ii) $\beta(t)\ (\log(k/t))^{1+\varepsilon} \in \mathrm{BV}(0,\pi)$, *where* $\varepsilon>0$ *and* $k \geqslant \pi e^2$, *then* $\sum_{n=1}^{\infty} A_n(x)$ *is summable* $|\mathrm{R}, \exp(n^\alpha), 1|$ $(0<\alpha<1)$.

Theorem 2. *If* (i) *and* (ii) *of Theorem* 1 *hold and* $\{n^\alpha A_n(x)\} \in \mathrm{BV}$, *for* $0<\alpha<1$, *then* $\sum_{n=1}^{\infty} |A_n(x)| < \infty$.

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(1) MOHANTY [3].

(2) HARDY and LITTLEWOOD [1], [2].

(3) PATI [6].

3. – We shall require the following order-estimates for the proof of our theorems:

(3.1) $$\sum_{\exp(n^\alpha)\leqslant w} \exp(n^\alpha)/n = O(w/\log w) \quad (^1)\,,$$

(3.2) $$K(w,t) = O\big(w/(\log w)^{1/\alpha}\, t\big) \quad (^2)\,,$$

(3.3) $$R(n,t) = O\{(\log(n+1))^{-p}\}\,,$$

(3.4) $$\eta(w) = O\{w/\log w\,(\log\,\log w)^p\}\,.$$

Proof of (3.3). Case (i). When $n_1^{-1}\leqslant t$, where $n_1 = n+1$, we have

$$R(n,t) = \Big(\int_0^{1/n_1} + \int_{1/n_1}^{t}\Big)(\sin nu)\Big/\Big\{u\Big(\log\frac{k}{u}\Big)^p\Big\}\cdot du = J_1 + J_2\,, \qquad \text{say}\,.$$

By the second mean value theorem, we have

$$J_1 = (\log kn_1)^{-p}\int_\eta^{1/n_1}\frac{\sin nu}{u}\,du \qquad (0<\eta<n_1^{-1})$$
$$= O\{(\log n_1)^{-p}\}\,.$$

Now, since $u^{-1}\,(\log(k/u))^{-p}$ is decreasing in (n_1^{-1}, t), we have, again by the second mean value theorem,

$$J_2 = n_1(\log kn_1)^{-p}\int_{1/n_1}^{t'}\sin nu\,du \qquad (n_1^{-1}<t'<t)$$
$$= O\{(\log n_1)^{-p}\}\,.$$

Case (ii). When $n_1^{-1}>t$, for the n_1 defined in Case (i), we have

$$R(n,t) = \Big(\int_0^{1/n_1} - \int_t^{1/n_1}\Big)(\sin nu)\Big/\Big\{u\Big(\log\frac{k}{u}\Big)^p\Big\}\cdot du = J_1 + J_2'\,, \qquad \text{say}\,.$$

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(1) MOHANTY [3], (4.2).
(2) MOHANTY [3], (4.1).

Now, since $(\log (k/u))^{-p}$ is monotonic increasing in (t, n_1^{-1}), we have

$$|J_2'| = O\left\{(\log kn_1)^{-p} \int_t^{1/n_1} \frac{|\sin nu|}{u}\, du\right\} = O\{(\log n_1)^{-p}\}\,.$$

Hence, finally, it follows that

$$R(n, t) = O\{(\log (n+1))^{-p}\}\,.$$

Proof of (3.4). Let $\exp(m^\alpha) \leqslant w < \exp\{(m+1)^\alpha\}$, then

$$\eta(w) = \sum_{n=1}^{m} \exp(\alpha^n)/n(\log (n+1))^p = \sum_{n=1}^{q-1} \ldots + \sum_{n=q}^{m} \ldots = P + Q, \qquad \text{say,}$$

where q is so chosen that

(i) $$\exp(n^\alpha)/n(\log (n+1))^p$$

is steadily increasing for $n \geqslant q$ and

(ii) $$1 - \left(q^{-\alpha} + p/\alpha q^\alpha \log (1+q)\right) \geqslant \varDelta > 0$$

for strictly positive number $\varDelta$.

Now, $P = O(1)$, and

$$Q < \int_q^{m+1} \exp(x^\alpha)\, x^{-1} \left(\log (x+1)\right)^{-p} dx = J, \qquad \text{say}\,.$$

Now

$$J = \frac{1}{\alpha} \exp(x^\alpha)\, x^{-\alpha} \left(\log (x+1)\right)^{-p} +$$
$$+ \int_q^{m+1} \exp(x^\alpha)\, x^{-1} \left(\log (x+1)\right)^{-p} \left(x^{-\alpha} + px^{1-} /\alpha(1+x) \log (1+x)\right) dx\,.$$

Therefore

$$J < \frac{\exp\{(m+1)^\alpha\}\,(m+1)^{-\alpha}}{\alpha(\log (m+2))^p} + \left(q^{-\alpha} + p/\alpha\, q^\alpha \log (1+q)\right) J\,,$$

so that

$$J = O\{w(\log w)^{-1}\ (\log \log w)^{-p}\}\,.$$

4. – For the proof of the theorems we require the following lemmas:

Lemma 1 [1]. If (i) $\sum_{n=1}^{\infty} a_n$ is summable $|R, \lambda, k|$ $(k > 0)$, (ii) $\{a_n \lambda_n/(\lambda_n - \lambda_{n-1})\}$ $\in$ BV and (iii) $\{\lambda_n/\lambda_{n+1}\} \in$ BV, then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

Lemma 2. *The integral*

$$I = \int_{e^2}^{\infty} w^{-2} \Big| \sum_{\exp(n^\alpha) \leqslant w} \exp(n^\alpha)\; g(n, \pi) \Big| \, dw \,,$$

is convergent.

Proof. Integrating by parts, we have

$$g(n, \pi) = -\frac{1}{n}\int_0^\pi (\sin nt) \Big/ \left\{t\left(\log\frac{k}{t}\right)^{1+\varepsilon}\right\} \cdot dt + \frac{1}{n}\int_0^\pi (\sin nt) \Big/ \left\{t\left(\log\frac{k}{t}\right)^{2+\varepsilon}\right\} \cdot dt$$

$$= O\{n^{-1}(\log(n+1))^{-1-\varepsilon}\} + O\{n^{-1}(\log(n+1))^{-2-\varepsilon}\}$$

(by (3.3))

$$= O\{n^{-1}(\log(n+1))^{-1-\varepsilon}\} \,.$$

Therefore by (3.4), we have

$$I = O\Big\{\int_{e^2}^{\infty} w^{-1}(\log w)^{-1}(\log\log w)^{-1-\varepsilon}\, dw\Big\} = O(1) \,.$$

Proof of Theorem 1. Since,

$$A_n(x) = \frac{2}{\pi}\int_0^\pi \varphi(t) \cos nt \, dt \,,$$

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(1) PATI [6].

we have, integrating by parts,

$$A_n(x) = -\frac{2}{\pi}\int_0^{\pi} \frac{\sin nt}{nt}\, t\, \mathrm{d}\varphi(t)$$

$$= \frac{2}{\pi}\int_0^{\pi} \beta(t)\, t\, \frac{\partial}{\partial t}\, \frac{\sin nt}{nt}\, \mathrm{d}t$$

$$= \frac{2}{\pi}\beta(\pi)\left(\log\frac{k}{\pi}\right)^{1+\varepsilon}\int_0^{\pi} u\left(\log\frac{k}{u}\right)^{-1-\varepsilon}\frac{\partial}{\partial u}\,\frac{\sin nu}{nu}\,\mathrm{d}u$$

$$-\frac{2}{\pi}\int_0^{\pi} d\left\{\beta(t)\left(\log\frac{k}{t}\right)^{1+\varepsilon}\right\}\int_0^{t}\frac{u}{(\log(k/u))^{1+\varepsilon}}\,\frac{\partial}{\partial u}\,\frac{\sin nu}{nu}\,\mathrm{d}u\,.$$

The series $\sum_{n=1}^{\infty} A_n(x)$ is summable $|\mathrm{R}, \exp(n^{\alpha}), 1|$, if

$$I = \int_{e^2}^{\infty} w^{-2}\Big|\sum_{\exp(n^{\alpha})\leqslant w}\exp(n^{\alpha})\, A_n(x)\Big|\,\mathrm{d}w < \infty\,.$$

For the proof of the Theorem, since by hypothesis $\beta(\pi)\,\log\,((k/\pi))^{1+\varepsilon}$ and $\int_0^{\pi}|\mathrm{d}\{\beta(t)\;(\log(k/t))^{1+\varepsilon}\}|$ are finite, it is sufficient to prove that

$$I_1 = \int_{e^2}^{\infty} w^{-2}\Big|\sum_{\exp(n^{\alpha})\leqslant w}\exp(n)\; g(n,\pi)\Big|\,\mathrm{d}w < \infty\,; \tag{5.1}$$

$$I_2 = \int_{e^2}^{\infty} w^{-2}\Big|\sum_{\exp(n^{\alpha})\leqslant w}\exp(n^{\alpha})\; g(n,t)\Big|\,\mathrm{d}w = O(1)\,, \tag{5.2}$$

uniformly in $0 < t < \pi$.

P r o o f of (5.1) This follows from Lemma 2.

P r o o f of (5.2). Integrating by parts, we have

$$g(n,t) = \frac{1}{n}\left(\log\frac{k}{t}\right)^{-1-\varepsilon}\sin nt\, -$$

$$-\frac{1}{n}\int_0^{t}(\sin nu)\Big/\left\{u\left(\log\frac{k}{u}\right)^{1+\varepsilon}\right\}\cdot\mathrm{d}u - \frac{1+\varepsilon}{n}\int_0^{t}(\sin nu)\Big/\left\{u\left(\log\frac{k}{u}\right)^{2+\varepsilon}\right\}\cdot\mathrm{d}u\,.$$

Therefore, for the proof of (5.2), it is sufficient to show that

$$(5.2)\ \text{(i)} \qquad I_{2,1} = \int_{e^2}^{\infty} w^{-2} \left| \sum_{\exp(n^\alpha) \leqslant w} \exp(n^\alpha)\ (\sin nt) \Big/ \left\{ n \left(\log \frac{k}{t}\right)^{1+\varepsilon} \right\} \right| dw = O(1),$$

$$(5.2)\ \text{(ii)} \qquad I_{2,2} = \int_{e^2}^{\infty} w^{-2} \left| \sum_{\exp(n^\alpha) \leqslant w} \exp(n^\alpha)\ R(n, t) \right| dw = O(1),$$

uniformly in $0 < t < \pi$.

P r o o f of (5.2) (i). Let $\tau = (k/t)^{\alpha/(1-\alpha)}$, we have

$$I_{2,1} = \int_{e^2}^{e^\tau} \dots + \int_{e^\tau}^{\infty} \dots = I_{2,1,1} + I_{2,1,2}, \qquad \text{say}.$$

Now

$$I_{2,1,1} = O\left\{ \left(\log \frac{k}{t}\right)^{-1-\varepsilon} \int_{e^2}^{e^\tau} w^{-2} \left| \sum_{\exp(n^\alpha) \leqslant w} n^{-1} \exp(n^\alpha) \right| dw \right\}$$

$$= O\left\{ \left(\log \frac{k}{t}\right)^{-1-\varepsilon} \int_{e^2}^{e^\tau} w^{-1} (\log w)^{-1}\, dw \right\} \qquad \text{(by (3.1))}$$

$$= O(1),$$

uniformly in $0 < t < \pi$. And by (3.2), we have

$$I_{2,1,2} = O\left\{ t^{-1} \left(\log \frac{k}{t}\right)^{-1-\varepsilon} \int_{e^\tau}^{\infty} w^{-1}\ (\log w)^{-(1/\alpha)}\, dw \right\}$$

$$= O\left\{ t^{-1} \left(\log \frac{k}{t}\right)^{-1-\varepsilon} \tau^{(\alpha-1)/\alpha} \right\} = O(1),$$

uniformly in $0 < t < \pi$.

Proof of (5.2) (ii). For $p > 1$, we have, by (3.3),

$$I_{2,2} = O\{ \int_{e^2}^{\infty} w^{-2} | \sum_{\exp(n^\alpha) \leqslant w} \exp(n^\alpha)\, n^{-1} (\log(n+1))^{-p} | dw \}$$

$$= O\{ \int_{e^2}^{\infty} w^{-1} (\log w)^{-1} (\log\log w)^{-p}\, dw \} \qquad (\text{by } (3.4))$$

$$= O(1),$$

uniformly in $0 < t < \pi$.

This completes the proof of Theorem 1.

6. - Proof of Theorem 2.

It has been observed by MOHANTY [**3**] that the sequence (i) $\{\exp(n^\alpha)/\exp\{(n+1)^\alpha\}\}$ and (ii) $\{n^{\alpha-1}\exp(n^\alpha)/(\exp(n^\alpha) - \exp\{(n-1)^\alpha\}\}$ are of BV and hence the conditions (ii) and (iii) of Lemma 1 are satisfied. Thus Theorem 2 now follows from Theorem 1, by virtue of Lemma 1.

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