Riv. Mat. Univ. Parma (2) 12 (1971), 229-236

ARUN KUMAR AGARWAL (*)

On the Order and Type of Entire Dirichlet Series. (**)

1. – Let $f(s)$ be an entire function defined by an everywhere absolutely convergent DIRICHLET series

$$\sum_{n=1}^{\infty} a_n \exp(s\,\lambda_n)\,, \tag{1.1}$$

where $\lambda_1 \geqslant 0$, $\lambda_n < \lambda_{n+1} \to \infty$ with n, $s = \sigma + it$ and

$$\limsup_{n\to\infty} \frac{\log n}{\lambda_n} = 0\,. \tag{1.2}$$

We shall use the notations

$$M(\sigma) = \underset{-\infty<t<\infty}{\text{l.u.b.}} |f(\sigma + it)|\,, \qquad \mu(\sigma = \max_{n\geqslant 0}(|a_n| \exp(\sigma\,\lambda_n))$$

and

$$\eta(\sigma) = \max_{n\geqslant 0}\{n \mid \mu(\sigma) = |a_n| \exp(\sigma\,\lambda_n)\}.$$

S. N. SRIVASTAVA ([1], [2]) has defined the mean value of $|f(s)|$ for $\text{Re}\,(s) = \sigma$ as

$$I_1(\sigma) = I_1(\sigma, f) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(\sigma + i\,t)|\,\mathrm{d}t \tag{1.3}$$

(*) Indirizzo: Department of Mathematics, Grambling College, Grambling, Louisiana 71245, U.S.A. .

(**) This work has been supported by a Senior Research Fellowship award of Council of Scientific and Industrial Research, New Delhi (India). — Ricevuto: 30-X-1970.

and

$$(1.4) \qquad m_{1,k}(\sigma) = m_{1,k}(\sigma, f) = \frac{2}{\exp(k\sigma)} \int_0^\sigma I_1(x) \exp(kx) \cdot dx ,$$

where k is any positive number. Also, he ([1], p. 133) has shown that $\log I_1(\sigma)$ is an increasing convex function of σ. This result enables us to express $\log I_1(\sigma)$ in the following form:

$$(1.5) \qquad \log I_1(\sigma) = \log I_1(\sigma_0) + \int_{\sigma_0}^{\sigma} L(x)\, dx ,$$

where $L(x)$ is an indefinitely increasing function, continuous in adjacent intervals.

It is known that for an entire function of linear order ϱ $(0 < \varrho < \infty)$

$$(1.6) \qquad \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log M(\sigma)}{\exp(\varrho\sigma)} = \begin{matrix}\tau\\ \nu\end{matrix} ,$$

where τ and ν are the type and lower type of $f(s)$ respectively. Finally, it is possible to find a function $\varrho(\sigma)$ ([3], p. 99) such that

(i) $\varrho(\sigma)$ is continuous for $\sigma > \sigma_0$ and $\limsup_{\sigma\to\infty} \varrho(\sigma) = \varrho$;

(ii) $\varrho(\sigma)$ is differentiable almost everywhere except at the end-points of adjacent intervals, where it possesses left-hand and right-hand derivatives;

(iii) $\limsup_{\sigma\to\infty} \{\exp(-\sigma\varrho(\sigma)) \log M(\sigma)\} = 1$;

(iv) $\limsup_{n\to\infty} \{\sigma\varrho'(\sigma)\} = 0$.

In this paper we have investigated a few relations with the help of (1.5). Also, we deduced a linear proximate order under certain conditions and the results obtained are given in the form of theorems.

2. – Theorem 1. *If $f(s)$ be an entire function of order ϱ $(0 < \varrho < \infty)$, type τ and lower type ν, then*

$$(2.1) \qquad \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log I_1(\sigma)}{\exp(\varrho\sigma)} = \begin{matrix}\tau\\ \nu\end{matrix} .$$

Proof. From (1.3), we have

$$\mu(\sigma) \leqslant I_1(\sigma) \leqslant M(\sigma) . \tag{2.2}$$

Also since

$$\log \mu(\sigma) \sim \log M(\sigma) ,$$

so with (2.2), we have

$$\log I_1(\sigma) \sim \log M(\sigma) . \tag{2.3}$$

The result follows from (1.6) and (2.3).

Theorem 2. *If $f(s)$ be an entire function of order ϱ $(0 < \varrho < \infty)$, type τ and lower type ν, then*

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log m_{1,k}(\sigma)}{\exp(\varrho\sigma)} = \begin{matrix}\tau\\ \nu\end{matrix} . \tag{2.4}$$

Proof. From (1.4), we have

$$m_{1,k}(\sigma) = 2\exp(-k\sigma)\int_0^\sigma I_1(x)\exp(kx)\,dx \leqslant (2/k)\,M(\sigma)\big(1-\exp(-k\sigma)\big) .$$

Taking limits, we get

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log m_{1,k}(\sigma)}{\exp(\varrho\sigma)} \leqslant \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log M(\sigma)}{\exp(\varrho\sigma)} . \tag{2.5}$$

Also, for $h \geqslant 0$ we have

$$m_{1,k}(\sigma+h) = 2\exp\big(-k(\sigma+h)\big)\int_0^{\sigma+h} I_1(x)\exp(kx)\,dx \geqslant (2/k)\big(1-\exp(-kh)\big)\,I_1(\sigma), \tag{2.6}$$

since $I_1(\sigma)$ is an increasing function of σ. Further using (2.2), we get

$$m_{1,k}(\sigma+h) \geqslant (2/k)\big(1-\exp(-kh)\big)\,\mu(\sigma) .$$

Taking limits, we get

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log m_{1,k}(\sigma)}{\exp(\varrho\sigma)} \geqslant \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \mu(\sigma)}{\exp(\varrho\sigma)} . \tag{2.7}$$

Now the result follows easily from (2.5) and (2.7), since for functions of finite order

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log M(\sigma)}{\exp(\varrho\,\sigma)} = \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \mu(\sigma)}{\exp(\varrho\,\sigma)} = \begin{matrix}\tau\\ \nu\end{matrix}.$$

3. – Let

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{V(\sigma)}{\exp(\varrho\,\sigma)} = \begin{matrix}\alpha\\ \beta\end{matrix}. \tag{3.1}$$

Theorem 3. *If $f(s)$ be an entire function of order ϱ, then*

$$\beta \leqslant \varrho\,\nu \leqslant \varrho\,\tau \leqslant \alpha .$$

Proof. From (1.5), for any $h \geqslant 0$,

$$\log I_1(\sigma + (h/\varrho)) = A_1 + \Big(\int_{\sigma_0}^{\sigma} + \int_{\sigma}^{\sigma+(h/\varrho)}\Big)\, V(x)\,\mathrm{d}x .$$

Let $0 < \beta < \infty$, then for any $\varepsilon > 0$ we have from (3.1)

$$\frac{V(\sigma)}{\exp(\varrho\,\sigma)} > \beta - \varepsilon \qquad \text{for } \sigma > \sigma_0 .$$

Hence

$$\frac{e^h \log I_1(\sigma+(h/\varrho))}{\exp[\varrho\,(\sigma+(h/\varrho))} > \frac{\beta-\varepsilon}{\varrho} + \frac{h}{\varrho}\,\frac{V(\sigma)}{\exp(\varrho\,\sigma)} + O(\exp(-\varrho\,\sigma)) .$$

Taking limits, we have

$$\nu\, e^h \geqslant \beta(1+h)/\varrho .$$

For $h = 0$, this gives

$$\nu\,\varrho \geqslant \beta .$$

Further, from (3.1) we have, for any $\varepsilon > 0$,

$$\frac{V(\sigma)}{\exp(\varrho\,\sigma)} < \alpha + \varepsilon \qquad \text{for } \sigma > \sigma_0 .$$

Hence

$$\log I_1(\sigma) = \int_{\sigma_0}^{\sigma} V(x)\,dx + A_2 < \frac{\alpha+\varepsilon}{\varrho}\exp(\varrho\sigma) + A_3 .$$

Taking limits, we get $\varrho\tau \leqslant \alpha$. This completes the proof of the Theorem 3.

Theorem 4. *Let $f(s)$ be an entire function of order ϱ $(0<\varrho<\infty)$, then*

$$\beta \leqslant (\alpha/e)\exp(\beta/\alpha) \leqslant \varrho\tau \leqslant \alpha$$

and

$$\beta \leqslant \varrho\nu \leqslant \beta\big(1+\log(\alpha/\beta)\big) \leqslant \alpha .$$

Proof. We have, for any $h \geqslant 0$,

$$\log I_1\big(\sigma + (h/\varrho)\big) = \int_{\sigma_0}^{\sigma+(h/\varrho)} V(x)\,dx + B_1 ,$$

hence

$$\exp\{h - \varrho(\sigma + (h/\varrho))\}\log I_1\big(\sigma+(h/\varrho)\big) >$$
$$> (\beta-\varepsilon)/\varrho + (h/\varrho)\exp(-\varrho\sigma).\,V(\sigma) + O\big(\exp(-\varrho\sigma)\big) .$$

Taking limits, we get

$$e^h\tau \geqslant (\beta + \alpha h)/\varrho \tag{3.2}$$

and

$$e^h\nu \geqslant \beta(1+h)/\varrho . \tag{3.3}$$

The maximum of right hand side of (3.2) occurs at $h = 1-(\beta/\alpha)$, substituting this value of h in (3.2), we get

$$e\varrho\tau \geqslant \alpha\exp(\beta/\alpha) \geqslant e\beta , \tag{3.4}$$

since, for $\xi \geqslant 0$,

$$e^{\xi}/\xi \geqslant e .$$

Again

$$\exp(-\varrho\sigma)\cdot\log I_1(\sigma+(h/\varrho)) < (\alpha+\varepsilon)/\varrho +$$

$$+ (h/\varrho)\exp(-\varrho\sigma)\cdot V(\sigma+(h/\varrho)) + O(\exp(-\varrho\sigma)) .$$

Taking limits, we get

$$e^h\tau \leqslant \alpha(1+he^h)/\varrho \tag{3.5}$$

and

$$e^h\nu \leqslant (\alpha+\beta h e^h)/\varrho . \tag{3.6}$$

The minimum of right hand side of (3.6) occurs at $h=\log(\alpha/\beta)$, substituting this value of h in (3.6), we get

$$\nu\varrho \leqslant \beta\big(1+\log(\alpha/\beta)\big) \leqslant \alpha . \tag{3.7}$$

Also putting $h=0$ in (3.3) and (3.5), respectively, we get

$$\nu\varrho \geqslant \beta \tag{3.8}$$

and

$$\tau\varrho \leqslant \alpha . \tag{3.9}$$

Combining (3.4) and (3.9), we get

$$\beta \leqslant (\alpha/e)\exp(\beta/\alpha) \leqslant \tau\varrho \leqslant \alpha;$$

and combining (3.7) and (3.8), we get

$$\beta \leqslant \nu\varrho \leqslant \beta\big(1+\log(\alpha/\beta)\big) \leqslant \alpha .$$

4. – Lemma. *Let $f(s)$ be an entire function of order ϱ and lower order λ, then*

$$\liminf_{\sigma\to\infty}\frac{I_1'(\sigma)}{I_1(\sigma)\log I_1(\sigma)} \leqslant \lambda \leqslant \varrho \leqslant \limsup_{\sigma\to\infty}\frac{I_1'(\sigma)}{I_1(\sigma)\log I_1(\sigma)} , \tag{4.1}$$

where $I_1'(\sigma)$ is the first derivative of $I_1(\sigma)$.

Proof. Let

$$\lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{I_1'(\sigma)}{I_1(\sigma)\log I_1(\sigma)} = \begin{matrix}A\\ B\end{matrix},$$

then, for any $\varepsilon > 0$, we have

$$\frac{I_1'(\sigma)}{I_1(\sigma)\log I_1(\sigma)} > \beta - \varepsilon \qquad \text{for } \sigma > \sigma_0 .$$

Integrating and taking limits, we get

$$\liminf_{\sigma\to\infty} \frac{I_1'(\sigma)}{I_1(\sigma)\log I_1(\sigma)} \leqslant \lambda .$$

Further, for any $\varepsilon > 0$,

$$\frac{I_1'(\sigma)}{I_1(\sigma)\log I_1(\sigma)} < \beta + \varepsilon \qquad \text{for } \sigma > \sigma_0 .$$

Integrating and taking limits, we get

$$\limsup_{\sigma\to\infty} \frac{I_1'(\sigma)}{I_1(\sigma)\log I_1(\sigma)} \geqslant \varrho .$$

Theorem 5. *Let $f(s)$ be an entire of order ϱ and lower order λ and satisfying* (4.1), *then the function*

$$\varrho(\sigma) = \big(\log\log I_1(\sigma)\big)/\sigma$$

is a proximate order of $f(s)$.

Proof. Consider the function

$$\varrho(\sigma) = \big(\log\log I_1(\sigma)\big)/\sigma ,$$

then $\lim_{\sigma\to\infty} \varrho(\sigma) = \varrho$, since $f(s)$ is of regular growth and of order ϱ. Further since ([1], p. 133) $\log I_1(\sigma)$ is an increasing convex function of σ, from this it follows that $\log I_1(\sigma)$ is differentiable almost everywhere with an increasing derivative; the set of points where the right hand derivative is greater than the left hand derivative is of measure zero. Consequently, $\varrho(\sigma)$ is differentiable almost everywhere. Thus

$$\varrho'(\sigma) = -(1/\sigma^2)\log\log I_1(\sigma) + (1/\sigma)\frac{I_1'(\sigma)}{I_1(\sigma)\log I_1(\sigma)},$$

so that

$$\sigma\varrho'(\sigma) = \left\{-\frac{\log\log I_1(\sigma)}{\sigma} + \frac{I_1'(\sigma)}{I_1(\sigma)\log I_1(\sigma)}\right\} + o(1)$$

which tends to zero as σ tends to infinity, since

$$\frac{I_1'(\sigma)}{I_1(\sigma)\log I_1(\sigma)}$$

and $\varrho(\sigma)$ have the same limit as $\sigma\to\infty$. Also, from the definition of $\varrho(\sigma)$,

$$\exp\left(-\sigma\varrho(\sigma)\right)\cdot\log I_1(\sigma) = 1$$

and since $\log I_1(\sigma)\sim\log M(\sigma)$ for functions of finite order, we obtain

$$\lim_{\sigma\to\infty}\left(\exp\left[-\sigma\varrho(\sigma)\right]\cdot\log M(\sigma)\right) = \lim_{\sigma\to\infty}\left(\exp\left[-\sigma\varrho(\sigma)\right]\cdot\log I_1(\sigma)\right) = 1\,.$$

Thus, we have seen that all the conditions for $\varrho(\sigma)$ to be a proximate order are satisfied and hence the Theorem 5 is proved.

References.

[1] S. N. Srivastava, *On the mean values of entire functions and their derivatives defined by Dirichlet series* (I), Rev. Mat. Hisp. Amer. (4) **27** (1967), 132-138.

[2] S. N. Srivastava, *On the mean values of entire functions defined by Dirichlet series* (to appear in Rev. Mat. Hisp. Ame.).

[3] R. P. Srivastava and R. K. Ghosh, *On entire functions represented by Dirichlet series*, Ann. Polon. Math. **13** (1963), 93-100.

* * *