Riv. Mat. Univ. Parma (2) 12 (1971), 179-187

K. L. SINGH (*)

# Nonexpansive Mappings and Fixed Point Theorems in Banach Spaces. (**)

## Introduction.

The basic problem of the theory of non-expansive mappings in BANACH spaces is quite similar to those of the general theory of non-linear mappings in linear topological spaces and center around the existence of fixed points, convergence of successive approximations, existence of proper extensions and so forth. Like many other theories of this kind, the theory of non-expansive mappings gives a good example of the paramount role played in the existence problems of analysis and topology by the compactness properties of sets and mappings. In the present paper few results related to nonexpansive mappings have been proved.

**1.1.** – A mapping $T\colon E \to E$ of a BANACH space $E$ into itself satisfying the following condition is said to be $s$-contraction,

$$\|Tx - Ty\| \leqslant s\|Tx - Ty\| + s\|x - y\| , \tag{1}$$

for all $x, y \in E$ and $0 < s < \frac{1}{2}$.

**2.1.** – Lemma. *Every s-contraction is one-to-one and continuous.*

Proof. $T$ is one-to-one, for if $x$ and $y$ are two points in $E$ such that $T(x) =$

(*) Indirizzo: Department of Mathematics, University of South Carolina, Columbia, South Carolina 29208, U.S.A..
(**) Ricevuto: 21-X-1969.

$= T(y)$, then by (1) we have

$$\|Tx - Ty\| \leqslant s\|Tx - Ty\| + s\|x - y\|$$

or

$$(1-s)\|Tx - Ty\| \leqslant s\|x - y\|$$

or

$$0 \leqslant s\|x - y\|, \qquad \text{since } 0 < s < \tfrac{1}{2}.$$

This implies

$$\|x - y\| = 0 \quad \text{or} \quad x = y.$$

To complete the proof of above Lemma, we need to show that $T$ is continuous at each point $x_0 \in E$. Let $\varepsilon_1 > 0$. Then

$$\|x - x_0\| < \varepsilon_1, \qquad \|Tx - Tx_0\| \leqslant s\|Tx - Tx_0\| + s\|x - x_0\|$$

i.e.

$$(1-s)\|Tx - Tx_0\| \leqslant s\|x - x_0\|$$

i.e.

$$\|Tx - Tx_0\| \leqslant \frac{s}{1-s}\|x - x_0\| < \varepsilon, \qquad \text{where} \qquad \varepsilon = \frac{s}{1-s}\varepsilon_1.$$

Thus, $T$ is continuous.

2.2. – Theorem. *Every s-contraction of a Banach space $E$ into itself has a unique fixed point.*

Proof. Let $x_0$ be an arbitrary point in $E$ and consider the sequence $\{x_n\}$. Let $x_1 = T(x_0)$, $x_2 = T(x_1) = T^2(x_0)$, ... . Now

$$\|x_{n+1} - x_n\| = \|T(x_n) - T(x_{n-1})\| \leqslant s\|T(x_n) - T(x_{n-1})\| + s\|x_n - x_{n-1}\|$$

or

$$(1-s)\|T(x_n) - T(x_{n-1})\| \leqslant s\|x_n - x_{n-1}\|$$

or

$$\|x_{n+1} - x_n\| = \|T(x_n) - T(x_{n-1})\| \leqslant \frac{s}{1-s}\|x_n - x_{n-1}\|$$

and

$$\|x_n - x_{n-1}\| = \|T[x_{n-1}) - T(x_{n-2})\| \leqslant s\|T(x_{n-1}) - T(x_{n-2})\| + s\|x_{n-1} - x_{n-2}\|$$

or

$$\|T(x_{n-1}) - T(x_{n-2})\| \leqslant \frac{s}{1-s}\|x_{n-1} - x_{n-2}\|.$$

Hence

$$\|x_{n+1} - x_n\| \leqslant \left(\frac{s}{1-s}\right)^2 \|x_{n-1} - x_{n-2}\|.$$

Therefore by continuing this process we have

$$\|x_{n+1} - x_n\| \leqslant \left(\frac{s}{1-s}\right)^n \|x_1 - x_0\| = \left(\frac{s}{1-s}\right)^n M, \qquad \text{where} \qquad M = \|x_1 - x_0\|.$$

Using this inequality we will show that the sequence $\{x_n\}$ is a CAUCHY sequence.

$$\|x_{n+p} - x_n\| \leqslant \|x_{n+p} - x_{n+p-1}\| + \|x_{n+p-1} - x_{n+p-2}\| + \dots + \|x_{n+1} - x_n\|$$

$$\leqslant M\left(\frac{s}{1-s}\right)^{n+p-1} + M\left(\frac{s}{1-s}\right)^{n+p-2} + \dots + M\left(\frac{s}{1-s}\right)^n$$

$$\leqslant M\left(\frac{s}{1-s}\right)^n \left[1 + \frac{s}{1-s} + \left(\frac{s}{1-s}\right)^2 + \left(\frac{s}{1-s}\right)^{p-1}\right]$$

$$\leqslant M\left(\frac{s}{1-s}\right)^n \frac{1-s}{1-2s}.$$

Since $0 < s < 1/2$, $M\left(\frac{s}{1-s}\right)^n \frac{1-s}{1-2s} \to 0$ as $n \to \infty$, and hence $\{x_n\}$ is a CAUCHY sequence.

Since $E$ is complete. Hence $\{x_n\}$ converges to some point $x_0 \in E$. Let $x_0 = T \lim_{n\to\infty} x_n$. Then by virtue of continuity of $T$, $Tx_0 = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = x_0$. Thus the existence of fixed point is proved.

Uniqueness. Assume that $x$ and $y$ are two fixed points of $T$, i.e. $T(x) = x$ and $T(y) = y$, $x \neq y$. Then since $T$ is $s$-contraction we have

$$\|x - y\| = \|Tx - Ty\| \leqslant s\|Tx - Ty\| + s\|x - y\|$$

or

$$(1-s)\|Tx-Ty\| \leqslant s\|x-y\| .$$

i.e.

$$(1-s)\|x-y\| \leqslant s\|x-y\|.$$

If $x \neq y$, we get $(1-s) \leqslant s$. But $0 < s < \frac{1}{2}$, hence a contradiction. Thus $x \neq y$.

2.3. – Theorem. *Let $T$ be an operator mapping a Banach space $E$ into itself. Suppose $T^k$ is s-contraction for some positive integer $k$. Then $T$ has a unique fixed point in $E$.*

Proof. Let $T^k = f$. Thus $f$ is a $s$-contraction of $E$ into itself, and by Theorem 2.2 has a unique fixed point. If $x_0$ is the fixed point of $f$, the relation $f(x_0) = x_0$ gives

$$T(f(x_0)) = T(x_0), \qquad \text{but} \qquad T^{k+1} = T(f) = f(T).$$

Therefore, $T(f(x_0)) = f(x_0)$.

Hence $f(x_0)$ is a fixed point of $T$, the uniqueness of this point shows that $T(x_0) = x_0$. In other words, $x_0$ is also a fixed point of $T$.

3.1. – An operator $T$ is said to be contraction on a domain $D$ if $T$ is defined on $D$ and there is a constant $\alpha$, $0 < \alpha < \frac{1}{2}$ such that

$$\|T(x_1) - T(x_2)\| \leqslant \alpha \|T(x_1) - x_1\| + \|(Tx_2) - x_2\|, \qquad \text{for any } x_1, x_2 \in D .$$

The infimum of all possible constants $\alpha$ that is

$$\operatorname*{Sup}_{x_1, x_2 \in D} \frac{\|Tx_1 - Tx_2\|}{\|Tx_1 - x_1\| + \|Tx_2 - x_2\|}$$

is called the contraction norm of $T$ on $D$ and is denoted by $\|T\|_D$. In sequel we will drop $D$ and we denote $\|T\|_D$ simply by $\|T\|$.

3.2. – Remark. In this case $T$ is not necessarily continuous. For example, let $X = [0, 1]$, $T(x) = x/4$ for $x \in [0, \frac{1}{2}]$, $T(x) = x/5$ for $x \in [\frac{1}{2}, 1]$ and the distance $d$ is the ordinary distance on the line. Now taking $\alpha = 4/9$ the above condition is satisfied and $T$ is discontinuous [2].

3.3. – Theorem. *Let $T$ be an operator with domain $K$ and range in a Banach space $E$. Suppose that there exists an $a_0 \in K$ such that $T$ is contraction on the closed sphere $\bar{S}_r(a)$: $\|a - a_0\| \leqslant r$ about the point $a$ which is contained in $K$, where*

$$r \geqslant \frac{\|\alpha_0 - Ta_0\|}{(1-2\alpha)(1-\alpha)^{n-1}}.$$

*Then $T$ has a unique fixed point in $\bar{S}_r(a)$.*

Proof. We show by induction that the sequence $\{a_n\}$ lies in $\bar{S}_r(a)$. If it is clear that $a_0$ and $a_1 = T(a_0)$ are in $\bar{S}_r(a)$. Suppose that $a_k$, $k > 1$, and $a_{k+1} \in \bar{S}_r(a)$. Then, for all $k$, $2 \leqslant k < n$,

$$\|a_k - a_{k+1}\| = \|T^k(a_0) - T^{k+1}(a_0)\| \leqslant \alpha\|T^k(a_0) - T^{k-1}(a_0)\| + \alpha\|T^{k+1}(a_0) - T^k(a_0)\|$$

or

$$(1-\alpha)\|T^k(a_0) - T^{k+1}(a_0)\| \leqslant \alpha\|T^k(a_0) - T^{k-1}(a_0)\|.$$

Hence

$$\|T^k(a_0) - T^{k+1}(a_0)\| \leqslant \frac{\alpha}{1-\alpha}\|T^k(a_0) - T^{k-1}(a_0)\|.$$

Now

$$\|T^k(a_0) - T^{k+1}(a_0)\| \leqslant \alpha\|T^{k-1}(a_0) - T^{k-2}(a_0)\| + \alpha\|T^k(a_0) - T^{k-1}(a_0)\|$$

or

$$\|T^k(a_0) - T^{k-1}(a_0)\| \leqslant \frac{\alpha}{1-\alpha}\|T^{k-1}(a_0) - T^{k-2}(a_0)\|.$$

Therefore

$$\|a_k - a_{k+1}\| \leqslant \left(\frac{\alpha}{1-\alpha}\right)^2 \|T^{k-1}(a_0) - T^{k+2}(a_0)\| = \left(\frac{\alpha}{1-\alpha}\right)^2 \|a_{k-2} - a_{k-1}\|.$$

Continuing this process we have

$$\|a_k - a_{k+1}\| \leqslant \sum_{k=2}^{n} \|a_k - a_{k+1}\|$$

$$\leqslant \sum_{k=2}^{n}\left(\frac{\alpha}{1-\alpha}\right)^{k-1}\|a_0 - a_1\| = \sum_{k=2}^{n}\left(\frac{\alpha}{1-\alpha}\right)^{k-1}\|a_0 - Ta_0\| \leqslant \frac{1-\alpha^n}{(1-2\alpha)(1-\alpha)^{n-1}}\|a_0 - T(a_0)\|.$$

But

$$r > \frac{\|a_0 - T(a_0)\|}{(1-2\alpha)(1-\alpha)^{n-1}}$$

or

$$(1-2\alpha)(1-\alpha)^{n-1}\, r > \|a_0 - T(a_0)\| \, .$$

Hence

$$\|a_k - a_{k+1}\| \leqslant \frac{1-\alpha^n}{(1-2\alpha)(1-\alpha)^{n-1}}\, r\,(1-2\alpha)(1-a)^{n-1} = (1-\alpha^n)\; r < r \quad \text{as} \quad \alpha < \tfrac{1}{2}\, .$$

Thus $a_{n+1} \in \bar{S}_r(a)$. The Theorm now follows from theorem 1 [**2**]. Since $a_n$ is contained in $\bar{S}_r(a)$ which is complete being a closed subject of a BANACH space and $T$ is contraction on $\bar{S}_r(a)$.

The above Theorem is a local version of Theorem 1 [**2**].

**3.4.** – Theorem. *Let $T$ be an operator satisfying the condition*

$$\|T(x) - T(y)\| \leqslant M\|T(x) - T(y)\| + M\|x - y\|, \qquad x, y \in \bar{S}_r(x_0)\, ,$$

*where $x_0$ is some fixed element in the Banach space $E$, $r > 0$. Then the equation*

$$x - \lambda\, Tx = f\, ,$$

*where $|\lambda| < \frac{1}{2}M$ has a unique solution $x^*$ in $\bar{S}_r(x_0)$ for each $f$ for which*

$$\|f + \lambda\, Tx_0 - x_0\| \leqslant (1 - M\,|\,\lambda\,|)\;\, r - M\|\lambda\| r\, Tx - Tx_0\|\, .$$

*The sequence $x_n = f + \lambda\, Tx_{n-1}$ converges to $x^*$ starting from the initial approximation $x_0$.*

Proof. Let $Kx = f + \lambda\, Tx$. Under the given condition of the Theorem, if $x \in \bar{S}_r(x_0)$, then

$$\begin{aligned}
\|Kx_0 - x_0\| &= \|f + \lambda\, Tx - x_0\| \\
&= \|f + \lambda\, Tx - \lambda\, Tx_0 + \lambda\, Tx_0 - x_0\| \\
&\leqslant \|\lambda\, Tx - \lambda\, Tx_0\| + \|f + \lambda\, Tx_0 - x_0\| \\
&\leqslant |\lambda|\, M\|Tx - Tx_0\| + |\lambda|\, M\|x - x_0\| + \|f + \lambda\, Tx_0 - x_0\|
\end{aligned}$$

$$\leqslant |\lambda|\, M\|Tx - Tx_0\| + |\lambda|\, M\, r + (1 - |\lambda|\, M)\, r - M\,|\lambda|\, \|Tx - Tx_0\| = r\, .$$

Thus $K$ maps $\bar{S}_r(x_0)$ into itself. Furthermore $K$ is a contraction on $\bar{S}_r(x_0)$, since for any $x_1, x_2 \in \bar{S}_r(x_0)$.

$$\begin{aligned}\|Kx_1 - Kx_2\| = \|f + \lambda Tx_1 - f - \lambda Tx_2\| &= \|\lambda Tx_1 - \lambda Tx_2\| \\ &\leqslant |\lambda| \, \|Tx_1 - Tx_2\| \\ &\leqslant |\lambda| M\{\|Tx_1 - Tx_2\| + \|x_1 - x_2\|\}\end{aligned}$$

and $M|\lambda| < \frac{1}{2}$, hence $K$ is a contraction. Thus the Theorem follows from the Theorem 2.2.

**4.1.** – We define a resolvent operator of $T$ as follows

$$R_\lambda T = \sum_{n=1}^{\infty} \lambda^{n-1} K^n T, \qquad |\lambda| \leqslant \frac{1}{\|K\|}.$$

We proved [5] the following

Theorem. *Let $T$ be an operator satisfying the condition*

$$\|Tx - Ty\| \leqslant \|x - y\| \in \bar{S}_r(x_0),$$

*where $x_0$ is some fixed element in the Banach space $E$, $r > 0$. Then the equation*

$$x - \lambda Tx = f,$$

*where $|\lambda| < 1$ has a unique solution $x^*$ in $\bar{S}_r(x_0)$ for each $f$ for which*

$$\|f + \lambda Tx_0 - x_0\| \leqslant (1 - |\lambda|) r.$$

The sequence $x_n = f + \lambda Tx_{n-1}$ converges to $x^*$ starting from the initial approximation $x_0$.

**4.2.** – Theorem. *Under the assumption of above theorem with $\varphi_0 = x$ and $Kx = x$, the resolvent operator satisfies the following properties*

1) $$(1 + |\lambda|)^{-1} \leqslant \frac{\|R_\lambda T_1 - R_\lambda T_2\|}{\|T_1 - T_2\|} \leqslant (1 - |\lambda|)^{-1}, \qquad T_1 \neq T_2;$$

2) *$R_\lambda$ is a continuous function of the parameter $\lambda$ for which $|\lambda| < 1$.*

Proof. By definition of $R_\lambda$, $R_\lambda T = K(R_\lambda T) + T$. Thus for any $T_1$ and

$T_2$ such that

$$\|T_1\| \leqslant (1-|\lambda|)r \quad \text{and} \quad \|T_2\| \leqslant (1-|\lambda|)r,$$

we have

$$\|R_\lambda T_1 - R\,T_2\| = \|\lambda K R_\lambda T_1 + T_1 - \lambda K R_\lambda T_2 - T_2\|$$

or

$$\|R_\lambda T_1 - R_\lambda T_2\| \leqslant |\lambda|\, \|K R_\lambda T_1 - K R_\lambda T_2\| + \|T_1 - T_2\|$$
$$\leqslant |\lambda|\, \|R_\lambda T_1 - R_\lambda T_2\| + \|T_1 - T_2\|$$

or

$$(1-|\lambda|)\, \|R_\lambda T_1 - R_\lambda T_2\| \leqslant \|T_1 - T_2\|$$

or

(1) $$\frac{\|R_\lambda T_1 - R_\lambda T_2\|}{\|T_1 - T_2\|} \leqslant (1-|\lambda|)^{-1}.$$

On the other hand

$$\|T_1 - T_2\| \leqslant \|R_\lambda T_1 - R_\lambda T_2\| + \|\lambda K R\, T_1 - \lambda K R_\lambda T_2\|$$
$$\leqslant \|R_\lambda T_1 - R_\lambda T_2\| + |\lambda|\, \|R_\lambda T_1 - R_\lambda T_2\|$$
$$\leqslant (1+|\lambda|)\, \|R_\lambda T_1 - RT_2\|.$$

Hence

(2) $$(1+|\lambda|)^{-1} \leqslant \frac{\|R_\lambda T_1 - R_\lambda T_2\|}{\|T_1 - T_2\|}.$$

Combining (1) and (2) we have

$$(1+|\lambda|)^{-1} \geqslant \frac{\|R_\lambda T_1 - R_\lambda T_2\|}{\|T_1 - T_2\|} \leqslant (1-|\lambda|)^{-1},$$

which completes the proof of 1).

To prove 2), let $\lambda$ and $\mu$ be two numbers such that $|\lambda| < 1$, $|\mu| < 1$. Then $R_\lambda$ and $R_\mu$ are defined for $T$ with $\|T\| = 1 - \max\{|\lambda|, |\mu|\}\, r$.

Then $(R_\lambda - R_\mu)T = [K R_\lambda T - K R_\mu] + (\lambda - \mu) K R_\mu f$. Hence

(3) $$\|(R_\lambda - R_\mu)T\| \leqslant |\lambda|\, \|R_\lambda T - R_\mu T\| + |\lambda - \mu|\, \|K R_\mu T\|.$$

Clearly $R_\mu x = x$ and from (1) we have

$$\|K R_\mu T\| = \|K R_\mu T - K R_\mu x\| \leqslant \frac{\|f\|}{1-|\mu|} . \tag{4}$$

Combining (3) and (4) we have

$$\|(R_\lambda R_\mu) T\| \leqslant \frac{\|f\|}{(1-|\mu|)(1-|\lambda|)} \|\lambda - \mu\|$$

and hence $R_\lambda$ is a continuous function of the parameter $\lambda$.

## References.


[1] J. Horváth, *Topological Vector Spaces and Distributions*, Vol. I, Addison-Wesley, London 1966.

[2] R. Kannan, *Some results on fixed points* (II), Amer. Math. Monthly **76** (1969) 405-408.

[3] M. Nashed, *Notes for nonlinear functional analysis course*, Georgia Inst. Tech. 1966 (unpublished).

[4] K. L. Singh, *Contractions mapping and fixed point theorems*, Ann. Soc. Sci. Bruxelles (1) **783** (1968), 34-44.

[5] K. L. Singh, *Nonexpansive mappings in Banach spaces*, J. Sci. Res. Banaras Hindu Univ. (to appear).

[6] E. H. Zarantonello, *Solving functional equations by contraction averaging*, M. R. C. Technical (Summary Report 160) (1960).


* * *