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Some Inequalities Related to Proximate Order. (**)

1. - Introduction.

A real valued function $\varrho(r)$ satisfying the conditions

$$(1.1) \qquad \begin{cases} \text{(i)} & \lim_{r\to\infty} \varrho(r) = \varrho \\ \text{(ii)} & \lim_{r\to\infty} [r\cdot\varrho'(r)\log r] = 0 \end{cases}$$

is called a proximate order ([1], p. 32). If for the entire function $f(z) = \sum_0^\infty a_n z^n$ the quantity

$$(1.2) \qquad \sigma_f = \overline{\lim_{r\to\infty}} \frac{\log M(r)}{r^{\varrho(r)}},$$

where $M(r) = \max_{|z|=r} |f(z)|$, is different from zero and infinity, then $\varrho(r)$ is called a proximate order of the given entire function $f(z)$, and σ_f is called the type of $f(z)$ with respect to proximate order $\varrho(r)$. Now let $\varphi(r)$ be a real valued positive function, finite on every finite interval and increasing to infinity monotonically. Let $\varrho(r)$ be a proximate order defined as above and

$$0 < \lim_{r\to\infty} \varrho(r) = \varrho < \infty.$$

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Let

(1.3) $$\psi(r) = \psi(r_0) + \int_{r_0}^{r} \varphi(t)\, t^{-\lambda}\, dt \qquad (\varrho+1 > \lambda \geqslant 1, \quad r_0 > 0)\,.$$

Let

(1.4) $$\begin{cases} \overline{\lim\limits_{r\to\infty}} \dfrac{\psi(r)\, r^{\lambda-1}}{r^{\varrho(r)}} = \dfrac{\alpha}{\beta} \\[2ex] \overline{\lim\limits_{r\to\infty}} \dfrac{\psi(r)\, r^{\lambda-1}}{\varphi(r)} = \dfrac{c}{d} \\[2ex] \overline{\lim\limits_{r\to\infty}} \dfrac{\varphi(r)}{r^{\varrho(r)}} = \dfrac{\nu}{\delta}\,. \end{cases}$$

In this paper we prove certain results involving the constant α, β; c, d and ν, δ togather with ϱ and λ which generalize some known results ([**1**] through [**9**)]. The remarks at the end of the proofs of the theorems illustrate the generalizations.

2. - Theorems.

Theorem 1. *Let ν, δ; c, d be defined as in* (1.4). *Then we have*

(2.1) $$\frac{\delta}{\nu}\, \frac{1}{\varrho + 1 - \lambda} \leqslant d \leqslant c \leqslant \frac{\nu}{\delta}\, \frac{1}{\varrho+1-\lambda} \qquad (0 < \delta \leqslant \nu < \infty)\,.$$

Proof. For the proximate order $\varrho(r)$, such that $\lim\limits_{r\to\infty} \varrho(r) = \varrho$ $(0<\varrho<\infty)$, the following properties ([**1**], pp. 33-34) are known:

(2.2a) $$\int_{\alpha}^{r} t^{\varrho(t)-\lambda}\, dt = \frac{1}{\varrho+1-\lambda}\, r^{\varrho(r)+1-\lambda} + o(r^{\varrho(r)+1-\lambda}) \qquad (\varrho + 1 > \lambda)$$

and

(2.2b) $$(1-\varepsilon)\, k^{\varrho}\, r^{\varrho(r)} < (k\,r)^{\varrho(kr)} < (1+\varepsilon)\, k^{\varrho}\, r^{\varrho(r)}$$

for $r \to \infty$ and uniformly in k for $0 < a \leqslant k \leqslant b < \infty$. By definition of ν and δ

we have, for given $\varepsilon > 0$,

$$\delta - \varepsilon < \frac{\varphi(r)}{r^{\varrho(r)}} < \nu + \varepsilon \qquad (r > r_0(\varepsilon)) . \tag{2.3}$$

Now

$$\psi(r) = \psi(r_0) + \int_{r_0}^{r} \varphi(t)\, t^{-\lambda}\, \mathrm{d}t < O(1) + (\nu + \varepsilon) \int_{r_0}^{r} t^{\varrho(t)-\lambda}\, \mathrm{d}t \qquad (r > r_0(\varepsilon)) . \tag{2.4}$$

Using (2.2a) we have

$$\psi(r) < O(1) + (\nu + \varepsilon) \left\{ \frac{1}{\varrho + 1 - \lambda} r^{\varrho(r)+1-\lambda} + o(r^{\varrho(r)+1-\lambda}) \right\} \qquad (r > r_0(\varepsilon)) .$$

or

$$\frac{\psi(r)\, r^{\lambda-1}}{\varphi(r)} < o(1) + \frac{\nu + \varepsilon}{\varrho + 1 - \lambda} \frac{r^{\varrho(r)}}{\varphi(r)} \qquad (r > r_0(\varepsilon)) .$$

Therefore

$$\overline{\lim_{r\to\infty}} \frac{\psi(r)\, r^{\lambda-1}}{\varphi(r)} = c \leqslant \frac{\nu + \varepsilon}{\varrho + 1 - \lambda} \overline{\lim_{r\to\infty}} \frac{r^{\varrho(r)}}{\varphi(r)} ,$$

or

$$c \leqslant \frac{\nu}{\delta} \frac{1}{\varrho + 1 - \lambda} . \tag{2.5}$$

Again, from (2.3) and (2.4),

$$\psi(r) > O(1) + (\delta - \varepsilon) \int_{r_0}^{r} t^{\varrho(t)-\lambda}\, \mathrm{d}t \qquad (r > r_0(\varepsilon))$$

$$= O(1) + (\delta - \varepsilon) \left\{ \frac{1}{\varrho + 1 - \lambda} r^{\varrho(r)+1-\lambda} + o(r^{\varrho(r)+1-\lambda}) \right\} ,$$

or

$$\underline{\lim_{r\to\infty}} \frac{\psi(r)\, r^{\lambda-1}}{\varphi(r)} = d \geqslant \frac{\delta}{\varrho + 1 - \lambda} \underline{\lim_{r\to\infty}} \frac{r^{\varrho(r)}}{\varphi(r)} ,$$

or

(2.6) $$d \geqslant \frac{\delta}{\nu}\,\frac{1}{\varrho+1-\lambda}\,.$$

Combining (2.5) and (2.6), we get (2.1).

Remark 1. Let $f(z)=\sum_0^\infty a_n z^n$ be an entire function of order ϱ, $0<\varrho<\infty$. Let $M(r)=\max_{|z|=r}|f(z)|$, $\mu(r)$ be the maximum term in TAYLOR expansion of $f(z)$ and $\nu(r)$ be the rank of maximum term $\mu(r)$.

Putting $\varphi(r)=\nu(r)$, $\lambda=1$ in (1.3), we get $\psi(r)\sim\log\mu(r)$. Further taking $\varrho(r)=\varrho$ in (1.4), we get the inequalities ([7], p. 22)

$$\delta/(\varrho\,\nu)\leqslant d\leqslant c\leqslant \nu/(\varrho\,\delta)$$

as a particular case of (2.1).

Theorem 2. *For the constants* $\alpha,\beta;\ \nu,\delta$ *as defined in* (1.4), *we have*

(2.7) $$\alpha\leqslant\frac{\nu}{\varrho+1-\lambda}\,k^{(\lambda-1-\varrho)/\varrho}+\frac{\nu}{\lambda-1}\,[k^{(\lambda-1)/\varrho}-1]\,,$$

(2.8) $$\beta\leqslant\frac{\nu}{\varrho+1-\lambda}\,k^{(\lambda-1-\varrho)/\varrho}+\frac{\delta}{\lambda-1}[k^{(\lambda-1)/\varrho}-1]$$

and

(2.9) $$\alpha\geqslant\frac{\delta}{\varrho+1-\lambda}\,k^{(\lambda-1-\varrho)/\varrho}+\frac{\nu}{k\,(\lambda-1)}\,[k^{(\lambda-1)/\varrho}-1]\,,$$

(2.10) $$\beta\geqslant\frac{\delta}{\varrho+1-\lambda}\,k^{(\lambda-1-\varrho)/\varrho}+\frac{\delta}{k\,(\lambda-1)}\,[k^{(\lambda-1)/\varrho}-1]\,,$$

for $\lambda>1$ *and* $k\geqslant 1$. *For* $\lambda=1$ *we have*

(2.7a) $$\delta\leqslant\varrho\beta\leqslant\delta\left(1+\log\frac{\nu}{\delta}\right)\leqslant\nu\,,$$

(2.7b) $$\delta\leqslant(\nu/e)\exp(\delta/\nu)\leqslant\varrho\,\alpha\leqslant\nu\,.$$

Further, for $\lambda \geqslant 1$ we have

$$\delta/(\varrho+1-\lambda) \leqslant \beta \leqslant \alpha \leqslant \nu/(\varrho+1-\lambda) . \tag{2.11}$$

Proof. First we take $\lambda > 1$. Let $k \geqslant 1$ be any constant. Consider

$$\psi(k^{1/\varrho} r) = \psi(r_0) + \int_{r_0}^{k^{1/\varrho} r} \varphi(t)\, t^{-\lambda}\, dt = O(1) + \int_{r_0}^{r} \varphi(t)\, t^{-\lambda}\, dt + \int_{r}^{k^{1/\varrho} r} \varphi(t)\, t^{-\lambda}\, dt . \tag{2.12}$$

Since $\varphi(t)$ is increasing, from (2.3) and (2.12) we have

$$\psi(k^{1/\varrho} r) < O(1) + (\nu+\varepsilon) \int_{r_0}^{r} t^{\varrho(t)-\lambda}\, dt + \varphi(k^{1/\varrho} r) \int_{r}^{k^{1/\varrho} r} t^{-\lambda}\, dt .$$

Using (2.2a) and (2.2b) ,

$$\frac{\psi(k^{1/\varrho} r)(k^{1/\varrho} r)^{\lambda-1}}{(k^{1/\varrho} r)^{\varrho(k^{1/\varrho} r)}} < O(1) + \frac{\nu+\varepsilon}{\varrho+1-\lambda}\, \frac{k^{(\lambda-1)/\varrho}}{(1-\eta)\, k} + \frac{\varphi(k^{1/\varrho} r)}{(k^{1/\varrho} r)^{\varrho(k^{1/\varrho} r)}}\, \frac{1}{\lambda-1} [k^{(\lambda-1)/\varrho} - 1]$$

for all large r. Hence

$$\alpha \leqslant \frac{\nu}{\varrho+1-\lambda} k^{(\lambda-1-\varrho)/\varrho} + \frac{\nu}{\lambda-1} [k^{(\lambda-1)/\varrho} - 1] ,$$

$$\beta \leqslant \frac{\nu}{\varrho+1-\lambda} k^{(\lambda-1-\varrho)/\varrho} + \frac{\delta}{\lambda-1} [k^{(\lambda-1)/\varrho} - 1] ,$$

and we get (2.7) and (2.8). Again from (2.3) and (2.12), we have

$$\psi(k^{1/\varrho} r) > O(1) + (\delta-\varepsilon) \int_{r_0}^{r} t^{\varrho(t)-\lambda}\, dt + \varphi(r) \int_{r}^{k^{1/\varrho} r} t^{-\lambda}\, dt .$$

Using (2.2a) and (2.2b), we get

$$\frac{\psi(k^{1/\varrho} r)(k^{1/\varrho} r)^{\lambda-1}}{(k^{1/\varrho} r)^{\varrho(k^{1/\varrho} r)}} >$$

$$> O(1) + \frac{\delta-\varepsilon}{\varrho+1-\lambda}\, \frac{k^{(\lambda-1)/\varrho}}{(1+\eta)\, k} + \frac{1}{\lambda-1}\, \frac{\varphi(r)}{(1+\eta)\, k\, r^{\varrho(r)}} [k^{(\lambda-1)/\varrho} - 1] \qquad \text{for all large } r.$$

Therefore

$$\alpha \geqslant \frac{\delta}{\varrho+1-\lambda}\, k^{(\lambda-1-\varrho)/\varrho} + \frac{\nu}{(\lambda-1)\,k}\,[k^{(\lambda-1)/\varrho}-1]$$

and

$$\beta \geqslant \frac{\delta}{\varrho+1-\lambda}\, k^{(\lambda-1-\varrho)/\varrho} + \frac{\delta}{(\lambda-1)\,k}\,[k^{(\lambda-1)/\varrho}-1]\,.$$

Thus we get (2.9) and (2.10).

For the case $\lambda=1$ we have, from (2.3) and (2.12),

$$\psi(k^{1/\varrho}\, r) < O(1) + (\nu+\varepsilon)\int_{r_0}^{r} t^{\varrho(t)-1}\,\mathrm{d}t + \varphi(k^{1/\varrho} r)\int_{r}^{k^{1/\varrho} r} t^{-1}\,\mathrm{d}t \qquad (r > r_0(\varepsilon))\,.$$

Using (2.2a) and (2.2b) ,we get

$$\frac{\psi(k^{1/\varrho}\, r)}{(k^{1/\varrho}\, r)^{\varrho(k^{1/\varrho}\, r)}} <$$

$$< O(1) + \frac{\nu+\varepsilon}{\varrho}\,\frac{r^{\varrho(r)}}{(1-\eta)\,k\,r^{\varrho(r)}} + \frac{\varphi(k^{1/\varrho}\, r)}{(k^{1/\varrho}\, r)^{\varrho(k^{1/\varrho}\, r)}}\,\frac{\log k}{\varrho} \qquad \text{for all large } r\,.$$

Therefore

$$\alpha \leqslant \frac{\nu}{k\,\varrho} + \frac{\nu \log k}{\varrho}\,, \qquad \beta \leqslant \frac{\nu}{k\,\varrho} + \frac{\delta \log k}{\varrho}\,. \tag{2.13}$$

Since k was arbitrary, taking $k=1$ in first and $k=\nu/\delta$ in the second of above inequalities, we get

$$\alpha \leqslant \nu/\varrho \qquad \text{and} \qquad \beta \leqslant (\delta/\varrho)\big(1+\log(\nu/\delta)\big)\,. \tag{2.14}$$

Again, we have

$$\psi(k^{1/\varrho} r) > O(1) + (\delta-\varepsilon)\int_{r_0}^{r} t^{\varrho(t)-1}\,\mathrm{d}t + \varphi(r)\int_{r}^{k^{1/\varrho} r} t^{-1}\,\mathrm{d}t\,.$$

So

$$\frac{\psi(k^{1/\varrho}\, r)}{(k^{1/\varrho}\, r)^{\varrho(k^{1/\varrho}\, r)}} >$$

$$> o(1) + + \frac{\delta-\varepsilon}{\varrho}\,\frac{r^{\varrho(r)}}{(1+\eta)\,k\,r^{\varrho(r)}} + \frac{\varphi(r)}{(1+\eta)\,k\,r^{\varrho(r)}}\,\frac{\log k}{\varrho} \qquad \text{for all large } r\,.$$

Therefore

$$\alpha \geqslant \frac{\delta}{\varrho k} + \frac{\nu}{\varrho}\,\frac{\log k}{k}\,, \qquad \beta \geqslant \frac{\delta}{\varrho k} + \frac{\delta}{\varrho}\,\frac{\log k}{k}\,. \tag{2.15}$$

Again taking $\log k = (\nu - \delta)/\nu$ in first and $k = 1$ in second of the above inequalities, we obtain

$$\alpha \geqslant \frac{\nu}{e\,\varrho}\exp(\delta/\nu)\,, \qquad \beta \geqslant \frac{\delta}{\varrho}\,. \tag{2.16}$$

Since $ex \leqslant e^x$ for $x \geqslant 0$; we have

$$e\,\nu/\delta \leqslant \exp(\nu/\delta) \qquad \text{or} \qquad 1 + \log(\nu/\delta) \leqslant \nu/\delta$$

and

$$e\,\delta/\nu \leqslant \exp(\delta/\nu) \qquad \text{or} \qquad \delta \leqslant (\nu/e)\exp(\delta/\nu)\,.$$

Combining (2.14) and (2.16), we get (2.7a) and (2.7b). For $\lambda > 1$, putting $k = 1$ in (2.7) and (2.10), we have

$$\delta/(\varrho + 1 - \lambda) \leqslant \beta \leqslant \alpha \leqslant \nu/(\varrho + 1 - \lambda)\,.$$

For $\lambda = 1$ we get, from (2.7a) and (2.7b), $\delta \leqslant \varrho\beta \leqslant \varrho\alpha \leqslant \nu$ and (2.11) follows.

Remarks. Let $n(r)$ denote the number of zeros inside the cirlce $|z| = r$ of the entire function $f(z) = \sum_0^\infty a_n z^n$ of order ϱ $(0 < \varrho < \infty)$, $M(r)$, $\mu(r)$ and $\nu(r)$ being the same as under Remark 1.

(i) Taking $\varphi(r) = \nu(r)$ and $\lambda = 1$ in (1.3), $\psi(r) \sim \log \mu(r)$; putting $\varrho(r) = \varrho$ in (1.4) the results of S. M. Shah ([5], pp. 220-223) follow from (2.7a) and (2.7b) as particular cases.

(ii) Let $G(r)$ denote the geometric mean of $f(z)$ for $|z| = r$, that is

$$G(r) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \log|f(r\exp(i\theta))|\,\mathrm{d}\theta\right\}.$$

By Jensen's theorem

$$\log G(r) = \int_0^r (n(t)/t)\,\mathrm{d}t - \log|f(0)|\,.$$

Taking $f(0) \neq 0$, $\varphi(r) = n(r)$ and $\lambda = 1$ in (1.3), $\psi(r) \sim \log G(r)$. Further taking $\varrho(r) = \varrho$ in (1.4); theorem 2 ([8], pp. 87-98) follows at once from (2.11) as a particular case.

(iii) Let $A(r) = \max_{|z|=r} \operatorname{Re} f(z)$. Then, for large r, $\log A(r) \sim \log M(r)$. Taking $\varphi(r) = \nu(r)$, $\lambda = 1$ and $\varrho(r) = \varrho$, in (1.3) and (1.4) theorem 6 ([3], pp. 203-224) follows from (2.11), (2.15), (2.7a) and (2.7b).

(iv) Let $\varphi(r) = n(r)$, $\lambda = 1$ in (1.3). Then

$$\psi(r) = N(r) = \int_0^r \big(n(t)/t\big)\, dt .$$

From (2.7b) we get $\nu \exp(\delta/\nu) \leqslant e\varrho\alpha$. Since $N(r) \leqslant \log M(r)$, $\alpha \leqslant \sigma_f$ we have $\nu \exp(\delta/\nu) \leqslant e\varrho\sigma_f$ which is a refinement of the result of LEVIN ([1], p. 44). Also $\delta \leqslant \varrho\alpha \leqslant \varrho\sigma_f$.

(v) Taking $\varrho(r) = \varrho$ in (iv), from (2.7a) and (2.7b) we get as particular cases the results ([4], p. 16)

$$\delta \leqslant \varrho\alpha \leqslant \varrho T , \qquad \nu \exp(\delta/\nu) \leqslant e\varrho\alpha \leqslant e\varrho T ,$$

where

$$\overline{\underline{\lim}}_{r\to\infty} \frac{\log M(r)}{r^{\varrho}} = \frac{T}{t} .$$

(vi) Keeping $\varphi(r)$, λ and $\varrho(r)$ as in (v), we get, from (2.7*b*), $\delta \leqslant \varrho\beta \leqslant \varrho t$ which is a refinement of the result $\delta \leqslant \varrho t$ ([9], pp. 51-55).

3. – In this section we investigate the asymptotic behaviour of $\psi(r)$ and $\varphi(r)$ and their mutual dependence.

Theorem 3. *The constants α, β; ν, δ having the same meaning as in* (1.4), *we have*

(3.1) $$0 < \beta \leqslant \alpha < \infty \quad \textit{imply} \quad 0 < \delta \leqslant \nu < \infty$$

and conversely.

Further

(3.2) $$\psi(r)\, r^{\lambda-1} \sim \alpha r^{\varrho(r)} \quad \textit{imply} \quad \varphi(r) \sim \alpha(\varrho+1-\lambda)\, r^{\varrho(r)} \quad (r\to\infty)$$

and conversely for $0<\alpha<\infty$.

Proof. When $\delta>0$ and $\nu<\infty$, by (2.11) we have $\alpha<\infty$ and $\beta>0$. Conversely, let $\beta>0$ and $\alpha<\infty$. When $\lambda\geqslant 1$ then, from (2.9) and (2.15) respectively, it follows that $\nu<\infty$.

Also $\beta>0$ implies $\delta>0$. Since, for $\delta=0$ we have from (2.8)

$$k^{(\varrho+1-\lambda)/\varrho}\beta \leqslant \nu/(\varrho+1-\lambda) \qquad \text{for } \lambda>1$$

and from (2.18) we have $k\beta\leqslant\nu/\varrho$ for $\lambda=1$, which is a contradiction since k is arbitrary and $\nu<\infty$. Thus (3.1) follows.

To prove (3.2), first let

$$\varphi(r)\sim\alpha(\varrho+1-\lambda)\, r^{\varrho(r)} \qquad \text{or} \qquad \nu=\delta=\alpha(\varrho+1-\lambda)\,.$$

From (2.11) we have

$$\delta/(\varrho+1-\lambda)\leqslant\beta\leqslant\alpha\leqslant\nu/(\varrho+1-\lambda)$$

and so $$\alpha=\beta \qquad \text{i.e.} \qquad \psi(r)\, r^{\lambda-1}\sim\alpha r^{\varrho(r)}.$$

Conversely, suppose

$$\psi(r)\, r^{\lambda-1}\sim\alpha r^{\varrho(r)}.$$

Then we have

$$\psi(r)=O(1)+\int_{r_0}^{r}\varphi(t)\, t^{-\lambda}\, dt\,.$$

Let $0<\eta<1$. Since $\varphi(r)$ is increasing, for $r>r_0$ we get

$$\frac{\varphi(r)\,\eta r}{(r+\eta r)^{\lambda}} < \int_{r}^{r+\eta r}\varphi(t)\, t^{-\lambda}\, dt = \psi(r+\eta r)-\psi(r)\,.$$

So asymptotically

$$\frac{\varphi(r)\,\eta r}{(r+\eta r)^{\lambda}} < \alpha\{(r+\eta r)^{\varrho(r+\eta r)+1-\lambda} - r^{\varrho(r)+1-\lambda} + o(r^{\varrho(r)+1-\lambda})\}\,.$$

Using (2.2b), we get

$$\frac{\varphi(r)}{r^{\lambda-1}}\,\frac{\eta}{(1+\eta)^{\lambda}} < \alpha\{(1+\eta)^{\varrho+1-\lambda}\, r^{\varrho(r)+1-\lambda} - r^{\varrho(r)+1-\lambda} + o(r^{\varrho(r)+1-\lambda})\}$$

or

$$\frac{\varphi(r)}{r^{\varrho(r)}} < \alpha\,\frac{(1+\eta)^{\lambda}}{\eta}\{(1+\eta)^{\varrho+1-\lambda}-1\} + o(1)\,,$$

or

$$\nu \leqslant \frac{\alpha}{\eta}\{(1+\eta)^{\varrho+1}-(1+\eta)^{\lambda}\}\,.$$

Since η is arbitrary, $(1+\eta)^{\varrho+1} \sim 1+(\varrho+1)\eta$ etc. and we get

$$\nu \leqslant \alpha\,(\varrho+1-\lambda)\,.$$

Again,

$$\frac{\varphi(r)\,\eta\, r}{(r-\eta r)^{\lambda}} > \int_{r-\eta r}^{r} \frac{\varphi(t)}{t^{\lambda}}\,\mathrm{d}t = \psi(r)-\psi(r-\eta r) \sim$$

$$\sim \alpha\{r^{\varrho(r)+1-\lambda} - (r-\eta r)^{\varrho(r-\eta r)+1-\lambda} + o(r^{\varrho(r)+1-\lambda})\}\,.$$

Using (2.2b), again, be have

$$\frac{\varphi(r)\,\eta\, r}{(r-\eta r)^{\lambda}} > \alpha\{r^{\varrho(r)+1-\lambda} - (1-\eta)^{\varrho+1-\lambda}\, r^{\varrho(r)+1-\lambda} + o(r^{\varrho(r)+1-\lambda})\}$$

or

$$\frac{\varphi(r)}{r^{\varrho(r)}} > \alpha\,\frac{(1-\eta)^{\lambda}}{\eta}\{1-(1-\eta)^{\varrho+1-\lambda}\} + o(1) \qquad \text{for larger } r,$$

or

$$\delta \geqslant \frac{\alpha}{\eta}\{(1-\eta)^{\lambda}-(1-\eta)^{\varrho+1}\} = \alpha\,(\varrho+1-\lambda)\,,$$

or

$$\alpha(\varrho + 1 - \lambda) \leqslant \delta \leqslant \nu \leqslant \alpha(\varrho + 1 - \lambda),$$

or

$$\delta = \nu = \alpha(\varrho + 1 - \lambda)$$

which imply $\varphi(r) \sim \alpha(\varrho + 1 - \lambda) r^{\varrho(r)}$ and (3.2) follows.

Remarks. (i) Putting $\varphi(r) = \nu(r)$, $\lambda = 1$, we have $\psi(r) \sim \log \mu(r)$. Further let $\varrho(r) = \varrho$ in (1.4), then the results of S. M. SHAH ([6], pp. 254-257) follow as particular cases of (3.1) and (3.2).

(ii) Taking $\varphi(r) = n(r)$, $\lambda = 1$ in (1.3) and $\varrho(r) = \varrho$ in (1.4), we get theorem 4 ([8], pp. 87-98) as a special case of (3.2).

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