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On the Absolute Riesz Summability Factors of Infinite Series. (**)

1. – Let $0 \leqslant \mu_0 < \mu_1 < \ldots < \mu_n < \ldots \to \infty$, and let $\sum a_n$ be a given infinite series. Then $\sum a_n$ is said to be summable $|R, \mu_n, 1|$, if ([2], [6])

$$\sum_n \Delta \frac{1}{\mu_n} \cdot \Big| \sum_{m=1}^{n} \mu_m a_m \Big| < \infty , \tag{1.1}$$

where $\Delta(1/\mu_n) = (1/\mu_n) - (1/\mu_{n+1})$.

It is known ([1], [5]) that summability $|R, e^n, 1|$ is equivalent to summability $|C, 0|$, which itself is equivalent to absolute convergence.

The extension of this definition of absolute Riesz summability to the index k, where $k > 1$, is given by

$$\sum_n \Delta \left(\frac{1}{\mu_n}\right)^k \Big| \sum_{m=1}^{n} \mu_m a_m \Big|^k < \infty \tag{1.2}$$

which is same as the definition [7] when $\alpha = 1$ and $\lambda = \mu_n$, and $\sum a_n$ is said to be summable $|R, \mu_n, 1|_k$.

It is obvious that summability $|R, \mu_n, 1|$ and summability $|R, \mu_n, 1|_1$ are the same.

In this paper we shall be concerned with the type $\mu_n = \exp n^\alpha$.

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2. – In 1965, TRIPATHI [4] established the following

Theorem T. Let $\{\lambda_n\}$ be a convex sequence such that $\sum n^{-1}\lambda_n < \infty$. If

$$\frac{1}{n}\sum_{\nu=1}^{n}|T_\nu| = O(1)\,, \tag{2.1}$$

where

$$T_n = \frac{1}{n}\sum_{i=1}^{n} i\,a_i \tag{2.2}$$

as $n\to\infty$, then the series $\sum \lambda_n a_n/n^\alpha$ is summable $|\mathrm{R}, \exp n^\alpha, 1|$.

The object of this paper is to extend this to $|\mathrm{R}, \exp n^\alpha, 1|_k$ summability. We shall prove the following

Theorem. *Let* $\{\lambda_n\}$ *be a convex sequence such that* $\sum n^{-1}\lambda_n < \infty$. *If*

$$\sum_{\nu=1}^{n}|T_\nu|^k = O(n)\,, \tag{2.3}$$

where

$$T_n = \frac{1}{n}\sum_{i=1}^{n} i\,a_i \tag{2.4}$$

as $n\to\infty$, *then the series* $\sum \lambda_n a_n/n^\alpha$ *is summable* $|\mathrm{R}, \exp n^\alpha, 1|_k$ *for* $0<\alpha\leqslant 1$.

3. – The lemmas needed for the proof of our Theorem are collected below:

Lemma 1 ([3], lemmas 3 and 4). If $\{\lambda_n\}$ is a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent, then λ_n is non-negative and decreasing, $n\,\Delta\lambda_n = o(1)$ and $\lambda_n \log n = O(1)$, as $n\to\infty$.

Lemma 2. Under the same conditions as in Lemma 1,

$$n\,\Delta(\lambda_n)^k = o(1) \quad \text{and} \quad (\lambda_n)^k \log n = O(1) \qquad \text{as } n\to\infty\,.$$

Proof. Since $\{\lambda_n\}$ is a convex sequence, therefore $\{(\lambda_n)^k\}$ is also a convex sequence and

$$\sum \frac{(\lambda_n)^k}{n} < \infty\,, \tag{3.1}$$

consequently the proof immediately follows from Lemma 1.

Lemma 3 ([4], lemma 2). If $\{\lambda_n\}$ is a convex sequence such that $\sum n^{-1}\lambda_n$ is convergent and

$$\chi_n = \lambda_n/n^{1+\alpha} \tag{3.2}$$

then, as $n \to \infty$,

$$n^2 \Delta\chi_n = O(1)\,, \tag{3.3}$$

$$\sum_{m=1}^{n} m\,\Delta\chi_m = O(1) \tag{3.4}$$

and

$$\sum_{m=1}^{n} m^2 \Delta^2 \chi_m = O(1)\,. \tag{3.5}$$

4. - Proof of the Theorem. From (1.1) it is clear that for establishing the Theorem we have to show that

$$\sum_n \Delta\left(\frac{1}{\mu_n}\right)^k \cdot \Big|\sum_{m=1}^{n} \mu_m \chi_m m\, a_m\Big|^k < \infty\,, \tag{4.1}$$

where

$$\mu_n = \exp n^\alpha \tag{4.2}$$

and

$$\chi_n = \lambda_n/n^{1+\alpha}\,. \tag{4.3}$$

Without any loss of generality we can suppose that $a_0 = 0$.
Now applying ABEL's transformation, we have

$$\sum_{m=1}^{n} \mu_m \chi_m m\, a_m = \sum_{m=1}^{n-1} \Delta(\mu_m \chi_m) \sum_{i=1}^{m} i\, a_i + \mu_n \chi_n \sum_{i=1}^{n} i\, a_i$$

$$= \sum_{m=1}^{n-1} m\, T_m \mu_m \Delta\chi_m + \sum_{m=1}^{n-1} m\, T_m \chi_{m+1} \Delta\mu_m + \mu_n \chi_n n\, T_n\,,$$

so that

$$\sum_{m=1}^{n} \mu_m \chi_m m\, a_m = \textstyle\sum_1 + \sum_2 + \Omega\,, \qquad \text{say}\,. \tag{4.4}$$

Hence applying HÖLDER's inequality with indices k and k', where $1/k + 1/k' = 1$, we have:

$$\sum_{n=2}^{\nu} \Delta\left\{\frac{1}{\mu_n}\right\}^k |\Sigma_1|^k = \sum_{n=2}^{\nu} \Delta\left\{\frac{1}{\mu_n}\right\}^k \Big|\sum_{m=1}^{n-1} m\, T_m \mu_m \Delta\chi_m\Big|^k$$

$$\leqslant \sum_{n=2}^{\nu} \Delta\left\{\frac{1}{\mu_n}\right\}^k \Big[\sum_{m=1}^{n-1} m\,(\Delta\chi_m)(\mu_m)^k |T_m|^k\Big] \cdot \Big[\sum_{m=1}^{n-1} m\,\Delta\chi_m\Big]^{k-1}$$

$$= O(1) \sum_{n=2}^{\nu} \Delta\left\{\frac{1}{\mu_n}\right\}^k \Big[\sum_{m=1}^{n-1} m\,(\Delta\chi_m)(\mu_m)^k |T_m|^k\Big]$$

$$= O(1) \sum_{m=1}^{\nu-1} m\,(\Delta\chi_m)(\mu_m)^k |T_m|^k \sum_{n=m+1}^{\nu} \Delta\left\{\frac{1}{\mu_n}\right\}^k$$

$$= O(1) \sum_{m=1}^{\nu} m\,(\Delta\chi_m) |T_m|^k = O(1) \sum_{m=1}^{\nu-1} \Delta(m\,\Delta\chi_m) \sum_{i=1}^{m} |T_i|^k + O(1)\,\nu(\Delta\chi_\nu) \sum_{i=1}^{\nu} |T_i|^k$$

$$= O(1) \sum_{m=1}^{\nu-1} m^2\,(\Delta^2\chi_m) + O(1) \sum_{m=1}^{\nu-1} m(\Delta\chi_m) + O(1)\,\nu^2(\Delta\chi_\nu)\,,$$

so that

$$\sum_{n=2}^{\nu} \Delta\left\{\frac{1}{\mu_n}\right\}^k |\Sigma_1|^k = O(1) \qquad \text{as } \nu \to \infty\,. \tag{4.5}$$

Also similarly

$$\sum_{n=2}^{\nu} \Delta\left\{\frac{1}{\mu_n}\right\}^k |\Sigma_2|^k = \sum_{n=2}^{\nu} \Delta\left\{\frac{1}{\mu_n}\right\}^k \Big|\sum_{m=1}^{n-1} m\, T_m \chi_{m+1} \Delta\mu_m\Big|^k$$

$$\leqslant \sum_{n=2}^{\nu} \Delta\left\{\frac{1}{\mu_n}\right\}^k \Big[\sum_{m=1}^{n-1} m\chi_{m+1}(\Delta\mu_m)^k |T_m|^k\Big] \Big[\sum_{m=1}^{n-1} m\chi_{m+1}\Big]^{k-1}$$

$$= O(1) \sum_{n=2}^{\nu} \Delta\left\{\frac{1}{\mu_n}\right\}^k \Big[\sum_{m=1}^{n-1} m\chi_{m+1}(\Delta\mu_m)^k |T_m|^k\Big]$$

$$= O(1) \sum_{m=1}^{\nu-1} m\chi_{m+1}(\Delta\mu_m)^k |T_m|^k \sum_{n=m+1}^{\nu} \Delta\left\{\frac{1}{\mu_n}\right\}^k$$

$$= O(1)\sum_{m=1}^{\nu} m^{\alpha}\chi_m |T_m|^k$$

$$= O(1)\sum_{m=1}^{\nu-1} \Delta(m^{\alpha}\chi_m)\sum_{i=1}^{m}|T_i|^k + O(1)\,\nu^{\alpha}\chi_\nu \sum_{i=1}^{\nu}|T_i|^k$$

$$= O(1)\sum_{m=1}^{\nu-1} m^{\alpha}\chi_m + O(1)\sum_{m=1}^{\nu-1} m^{\alpha+1}\Delta\chi_m + O(\lambda_\nu)$$

$$= O(1)\sum_{m=1}^{\nu-1}\frac{\lambda_m}{m} + O(1)\sum_{m=1}^{\nu-1}\Delta\lambda_m + O\left(\frac{1}{\log\nu}\right),$$

therefore

$$\sum_{n=2}^{\nu}\Delta\left\{\frac{1}{\mu_n}\right\}^k |\Sigma_2|^k = O(1)\,. \tag{4.6}$$

Again

$$\sum_{n=2}^{\nu}\Delta\left\{\frac{1}{\mu_n}\right\}^k |\Omega|^k = \sum_{n=2}^{\nu}\left\{\frac{1}{\mu_n}\right\}^k |\mu_n\chi_n n T_n|^k$$

$$= O(1)\sum_{n=2}^{\nu} n^{\alpha-1+k}|\chi_n|^k\,|T_n|^k = O(1)\sum_{n=2}^{\nu}\frac{(\lambda_n)^k}{n}\,|T_n|^k$$

$$= O(1)\sum_{n=2}^{\nu-1}\Delta\left\{\frac{(\lambda_n)^k}{n}\right\}\sum_{i=1}^{n}|T_i|^k + O(1)\left\{\frac{(\lambda_\nu)^k}{\nu}\right\}\sum_{i=1}^{\nu}|T_i|^k$$

$$= O(1)\sum_{n=2}^{\nu-1}\Delta(\lambda_n)^k + O(1)\sum_{n=2}^{\nu-1}\frac{(\lambda_n)^k}{n} + O(1)\frac{(\lambda_\nu)^k}{\nu^{k-1}}\,,$$

therefore

$$\sum_{n=2}^{\nu}\Delta\left\{\frac{1}{\mu_n}\right\}^k |\Omega|^k = O(1)\,. \tag{4.7}$$

Combining the estimates in (4.4), (4.5), (4.6) and (4.7), we find that (4.1) holds.

This completes the proof of the Theorem.

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