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On the Geometric Means of Products of Integral Functions. (**)

1. – Let $f_1(z), \dots, f_m(z)$ be m integral functions of orders $\varrho_1, \dots, \varrho_m$ respectively and let

$$G(r,\ f_1 \dots f_m) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \log|f_1(re^{i\theta}) \dots f_m(re^{i\theta})|\,\mathrm{d}\theta\right\}, \tag{1.1}$$

$$G(r,\ f_1^{(n)} \dots f_m^{(n)}) = \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \log|f_1^{(n)}(re^{i\theta}) \dots f_m^{(n)}(re^{i\theta})|\,\mathrm{d}\theta\right\}, \tag{1.2}$$

$$g_\delta(r,\ f_1 \dots f_m) = \exp\left\{\frac{\delta+1}{2\pi r^{\delta+1}}\int_0^{r}\int_0^{2\pi} \log|f_1(xe^{i\theta}) \dots f_m(xe^{i\theta})|\, x^\delta\,\mathrm{d}x\,\mathrm{d}\theta\right\} \tag{1.3}$$

and

$$g_\delta(r,\ f_1^{(n)} \dots f_m^{(n)}) = \exp\left\{\frac{\delta+1}{2\pi r^{\delta+1}}\int_0^{r}\int_0^{2\pi} \log|f_1^{(n)}(xe^{i\theta}) \dots f_m^{(n)}(xe^{i\theta})|\, x^\delta\,\mathrm{d}x\,\mathrm{d}\theta\right\} \tag{1.4}$$

where $f_1^{(n)}(z), \dots, f_m^{(n)}(z)$ are the n-th derivatives of $f_1(z), \dots, f_m(z)$ respectively and $0 < \delta < +\infty$.

In this paper we have considered geometric means of products of m integral functions and have obtained some of their properties. The results are given in the form of theorems and their corollaries.

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2. – Theorem 1. *For the class of integral functions for which*

$$\lim_{r\to\infty}\frac{l_2\, g_\delta(r, f_1 \dots f_m)}{\log r} = +\infty ,$$

we have

$$\lim_{r\to\infty}\,\begin{matrix}\sup\\ \inf\end{matrix}\,\frac{l_3\, g_\delta(r, f_1 \dots f_m)}{\log r} = \frac{\log L_\delta}{\log l_\delta},$$

where

$$\lim_{r\to\infty}\,\begin{matrix}\sup\\ \inf\end{matrix}\left\{\frac{\log G(r, f_1 \dots f_m)}{\log g_\delta(r, f_1 \dots f_m)}\right\}^{1/\log r} = \frac{L_\delta}{l_\delta},$$

$l_2\, x = \log\log x$ *and* $l_3 x = \log\log\log x$.

In order to prove this Theorem, we prove the following two lemmas:

Lemma 1. $\log G(r, f_1 \dots f_m)$ *is a convex function of* $\log r$, $f_s(0) \neq 0$ *for* $s = 1, 2, \dots, m$.

Proof. Using JENSEN's formula in (1.1), we have

$$\log G(r, f_1 \dots f_m) = \log|f_1(0) \dots f_m(0)| + \int_0^r \{n(x, f_1) + \dots + n(x, f_m)\}\frac{dx}{x}$$

$$= \log G(r_0, f_1 \dots f_m) + \int_{r_0}^r \{n(x, f_1) + \dots + n(x, f_m)\}\frac{dx}{x}.$$

This gives

$$\frac{d \log G(r, f_1 \dots f_m)}{d \log r} = n(r, f_1) + \dots + n(r, f_m).$$

The right hand side is a non-decreasing function of r, since $n(r, f)$ is a non-decreasing function of r and tends to infinity as $r \to \infty$.

Lemma 2. $r^{\delta+1}\{\log G(r, f_1 \dots f_m)\}^2$ *is a convex function of*

$$r^{\delta+1}\log g_\delta(r, f_1 \dots f_m).$$

Proof. We have from (1.1) and (1.3)

$$\frac{d[r^{\delta+1}\{\log G(r, f_1 \dots f_m)\}^2]}{d[r^{\delta+1} \log g_\delta(r, f_1 \dots f_m)]} = \frac{\dfrac{d}{dr}[r^{\delta+1}\{\log G(r, f_1 \dots f_m)\}^2]}{\dfrac{d}{dr}[(\delta+1)\int_0^r \log G(x, f_1 \dots f_m) x^\delta dx]}$$

$$= \log G(r, f_1 \dots f_m) + \frac{2r\ G'(r, f_1 \dots f_m)}{(\delta+1) G(r, f_1 \dots f_m)},$$

which increases with r for large values of r, since, by Lemma 1, $\log\ G(r, f_1 \dots f_m)$ is a convex function of $\log r$.

Proof of Theorem 1. We have

$$\log\{r^{\delta+1} \log g_\delta(r, f_1 \dots f_m)\} = (\delta+1)\int_0^r \frac{\log G(x, f_1 \dots f_m)}{\log g_\delta(x, f_1 \dots f_m)} \frac{dx}{x},$$

since numerator on the right hand side is the differential coefficient of the denominator.

This gives

$$\log\{r^{\delta+1} \log g_\delta(r, f_1 \dots f_m)\} < O(1) + (\delta+1)\int_{r_0}^r (L_\delta+\varepsilon)^{\log x} \frac{dx}{x},$$

for any $\varepsilon > 0$ and $r > r_0 = r_0(\varepsilon)$.

Now we obtain

$$\log\{r^{\delta+1} \log g_\delta(r, f_1 \dots f_m)\} < O(1) + (\delta+1)\frac{(L_\delta+\varepsilon)^{\log r}}{\log(L_\delta+\varepsilon)}.$$

Taking logarithm on both the sides and proceeding to limits, we get

$$\limsup_{r\to\infty} \frac{l_3\ g_\delta(r, f_1 \dots f_m)}{\log r} \leqslant \log L_\delta,$$

since

$$\lim_{r\to\infty} \frac{l_2\ g_\delta(r, f_1 \dots f_m)}{\log r} = +\infty.$$

Further, using Lemma 2, we have

$$\log\{(2r)^{\delta+1}\log g_\delta(2r, f_1 \dots f_m)\} >$$

$$> (\delta+1)\int_r^{2r} \frac{(\log G(x, f_1 \dots f_m))^2}{\log g_\delta(x, f_1 \dots f_m)\ \log G(x, f_1 \dots f_m)}\ \frac{dx}{x}$$

$$> (\delta+1)\frac{(\log G(r, f_1 \dots f_m))^2}{\log g_\delta(r, f_1 \dots f_m)}\ \frac{\log 2}{\log G(2r, f_1 \dots f_m)}$$

$$> (\delta+1)\,(L_\delta - \varepsilon)^{\log r}\ \frac{\log G(r, f_1 \dots f_m)}{\log G(2r, f_1 \dots f_m)}\log 2\,,$$

for a sequence of values of r tending to infinity. Consequently,

$$\limsup_{r\to\infty}\frac{\mathrm{l}_3\ g_\delta(r, f_1 \dots f_m)}{\log r} \geqslant \log L_\delta\,.$$

In a similar manner we prove that

$$\liminf_{r\to\infty}\frac{\mathrm{l}_3\ g_\delta(r, f_1 \dots f_m)}{\log r} = \log l_\delta\,.$$

This proves Theorem 1.

Theorem 2. *If*

$$\limsup_{r\to\infty}\frac{\mathrm{l}_2\ G(r, f_1 \dots f_m)}{\log r} = A$$

and

$$\limsup_{r\to\infty}\frac{\mathrm{l}_2\ g_\delta(r, f_1 \dots f_m)}{\log r} = B\,,$$

then

$$A = B = \max(\varrho_1, \dots, \varrho_m)\,. \tag{2.1}$$

Proof. Let $M(r, f_1), \ldots, M(r, f_m)$ denote respectively the maximum moduli of $f_1(z), \ldots, f_m(z)$ for $|z| = r$, then (1.1) in view of lemma on p. 311 of [1] gives

$$\text{(2.2)} \quad \begin{cases} \log G(r, f_1 \ldots f_m) \leqslant \\ \leqslant \log\left\{\frac{1}{2\pi}\int_0^{2\pi} |f_1(re^{i\theta}) \ldots f_m(re^{i\theta})|\,d\theta\right\} \leqslant \log\{M(r, f_1) \ldots M(r, f_m)\}\,. \end{cases}$$

Again, let $f(z)$ be regular in $|z| \leqslant R$ and let $z = re^{i\theta}$, $0 \leqslant r < R$, then Poisson-Jensen formula gives

$$\log|f(z)| = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)\log|f(Re^{i\varphi})|}{R^2 - 2Rr\cos(\theta - \varphi) + r^2}\,d\varphi - \sum_\mu \log\left|\frac{R^2 - \bar{a}_\mu re^{i\theta}}{R(re^{i\theta} - a_\mu)}\right|,$$

where a_μ are the zeros of $f(z)$ inside the circle $|z| \leqslant R$. Since each term in $\sum$ is positive, for $f(z) = f_1(z) \ldots f_m(z)$, this yields

$$\log|f_1(z) \ldots f_m(z)| \leqslant \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)\log|f_1(Re^{i\varphi}) \ldots f_m(Re^{i\varphi})|}{R^2 - 2Rr\cos(\theta - \varphi) + r^2}\,d\varphi\,.$$

Choosing z in such a manner that

$$\log\{M(r, f_1)\,|f_2(re^{i\theta})| \ldots |f_m(re^{i\theta})|\} \quad \text{or}$$

$$\log\{|f_1(re^{i\theta})|\,M(r, f_2) \ldots |f_m(re^{i\theta})|\} \quad \text{or} \ldots \text{or}$$

$$\log\{|f_1(re^{i\theta})| \ldots |f_{m-1}(re^{i\theta})|\,M(r, f_m)\} \leqslant \frac{R + r}{R - r}\log G(R, f_1 \ldots f_m)\,,$$

respectively, as ϱ_1 or ϱ_2 or ... or $\varrho_m = \max(\varrho_1, \ldots, \varrho_m)$.

Taking R to be $2r$, this gives

$$\log G(2r, f_1 \ldots f_m) \geqslant \tfrac{1}{3}\log\{M(r, f_1)\,|f_2(re^{i\theta})| \ldots |f_m(re^{i\theta})|\} \quad \text{or}$$

$$\log\{|f_1(re^{i\theta})|\,M(r, f_2) \ldots |f_m(re^{i\theta})|\} \quad \text{or} \ldots \text{or}$$

$$\text{(2.3)} \qquad \log G(2r, f_1 \ldots f_m) \geqslant \log\{|f_1(re^{i\theta})| \ldots |f_{m-1}(re^{i\theta})|\,M(r, f_m)\}\,.$$

Taking logarithms on both the sides of (2.2) and (2.3), proceeding to limits and combining the results thus obtained, we get

$$A = \max(\varrho_1, \ldots, \varrho_m) .$$

Further, since $\log G(r, f_1 \ldots f_m)$ is an increasing function of r, we have

$$\log g_\delta(r, f_1 \ldots f_m) = \frac{\delta+1}{r^{\delta+1}} \int_0^r \log G(x, f_1 \ldots f_m)\, x^\delta \, \mathrm{d}x \leqslant \log G(r, f_1 \ldots f_m) ,$$

which leads to $B \leqslant A$. Also,

$$\log g_\delta(2r, f_1 \ldots f_m) = \frac{\delta+1}{(2r)^{\delta+1}} \int_0^{2r} \log G(x, f_1 \ldots f_m)\, x^\delta \, \mathrm{d}x$$

$$\geqslant \frac{\delta+1}{(2r)^{\delta+1}} \int_r^{2r} \log G(x, f_1 \ldots f_m)\, x^\delta \, \mathrm{d}x$$

$$\geqslant \frac{2^{\delta+1}-1}{2^{\delta+1}} \log G(r, f_1 \ldots f_m) ,$$

which leads to $B \geqslant A$. Hence $A = B = \max(\varrho_1, \ldots, \varrho_m)$.

3. – Let $f_1(z), \ldots, f_m(z)$ be integral functions of orders ϱ_k $(0 < \varrho_k < \infty;$ $k = 1, \ldots, m)$. Further, let us set

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{n(r, f_1) + \ldots + n(r, f_m)}{r^\varrho} = \begin{matrix}\alpha\\ \beta\end{matrix} , \tag{3.1}$$

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{G(r, f_1 \ldots f_m)}{r^\varrho} = \begin{matrix}a\\ b\end{matrix} \tag{3.2}$$

and

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{g_\delta(r, f_1 \ldots f_m)}{r^\varrho} = \begin{matrix}c\\ d\end{matrix} , \tag{3.3}$$

where $\varrho = \max(\varrho_1, \dots, \varrho_m)$. We prove here the following results:

Theorem 3. *We have*

$$\text{(i)} \qquad e\,\varrho\, b \leqslant \varrho\, a + e\,\beta\,,$$

$$\text{(ii)} \qquad \alpha + \varrho\, b \leqslant e\,\varrho\, a\,,$$

$$\text{(iii)} \qquad 2^{\frac{\varrho+\delta+1}{\delta+1}}\, d \leqslant 2^{\frac{\varrho}{\delta+1}}\, b + c$$

and

$$\text{(iv)} \qquad a + d \leqslant 2^{\frac{\varrho+\delta+1}{\delta+1}}\, c\,.$$

Proof. We know that

$$n(r\,e^{1/\varrho}, f_1) + \dots + n(r\,e^{1/\varrho}, f_m) \geqslant \varrho \int_r^{r\,e^{1/\varrho}} \{n(x, f_1) + \dots + n(x, f_m)\}\, \frac{\mathrm{d}x}{x}\,.$$

Additing $\varrho \log G(r, f_1 \dots f_m)$ on both the sides, we obtain

$$\text{(3.4)} \qquad \begin{cases} \varrho \log G(r, f_1 \dots f_m) + n(r\,e^{1/\varrho}, f_1) + \dots + n(r\,e^{1/\varrho}, f_m) \\ \qquad\qquad \geqslant \varrho \log G(r\,e^{1/\varrho}, f_1 \dots f_m)\,. \end{cases}$$

In a similar manner, we obtain

$$\text{(3.5)} \qquad \varrho \log G(r, f_1 \dots f_m) + n(r, f_1) + \dots + n(r, f_m) \leqslant \varrho \log G(r\,e^{1/\varrho}, f_1 \dots f_m)\,.$$

Dividing (3.4) and (3.5) by r^ϱ, proceeding to limits and using (3.1) and (3.2), the results (i) and (ii) follow. Further, we have

$$\log G(2^{1/(\delta+1)} r, f_1 \dots f_m) \geqslant \frac{\delta+1}{r^{\delta+1}} \int_r^{2^{1/(\delta+1)} r} \log G(x, f_1 \dots f_m)\, x^\delta\, \mathrm{d}x\,,$$

where $0 < \delta < \infty$.

Adding $\log g_\delta(r, f_1 \dots f_m)$ on both the sides, this gives

$$\text{(3.6)} \qquad \log g_\delta(r, f_1 \dots f_m) + \log G(2^{1/(\delta+1)} r, f_1 \dots f_m) \geqslant 2 \log g_\delta(2^{1/(\delta+1)} r, f_1 \dots f_m)\,.$$

Similarly, we obtain

$$\text{(3.7)} \qquad \log g_\delta(r, f_1 \dots f_m) + \log G(r, f_1 \dots f_m) \leqslant 2 \log g_\delta(2^{1/(\delta+1)} r, f_1 \dots f_m) .$$

Dividing (3.6) and (3.7) by r^ϱ, taking limits and using (3.2) and (3.3), the results (iii) and (iv) follow.

Theorem 4. *If $f_1(z), \dots, f_m(z)$ are m integral functions other than polynomials and if $f_s(0) \neq 0$ for $s = 1, 2, \dots, m$, then*

$$\liminf_{r\to\infty} \frac{\log g_\delta(r, f_1 \dots f_m)}{r^\varrho} \geqslant \frac{\beta(\delta+1)}{\varrho(\varrho+\delta+1)}$$

and

$$\limsup_{r\to\infty} \frac{\log g_\delta(r, f_1 \dots f_m)}{r^\varrho} \leqslant \frac{\alpha(\delta+1)}{\varrho(\varrho+\delta+1)},$$

where $\varrho = \max(\varrho_1, \dots, \varrho_m)$.

Proof. Using JENSEN's formula in (1.1), we have

$$\text{(3.8)} \quad \log G(r, f_1 \dots f_m) = \log G(r_1, f_1 \dots f_m) + \int_{r_1}^{r} \{n(x, f_1) + \dots + n(x, f_m)\} \frac{dx}{x} .$$

From (3.1), we have for any $\varepsilon > 0$ and $r > r_2 = r_2(\varepsilon)$,

$$n(r, f_1) + \dots + n(r, f_m) > (\beta - \varepsilon) r^\varrho .$$

Therefore, from (3.8)

$$\text{(3.9)} \qquad \log G(r, f_1 \dots f_m) > \log G(r_1, f_1 \dots f_m) + (\beta - \varepsilon) \frac{r^\varrho - r_1^\varrho}{\varrho}, \qquad (r_1 \geqslant r_2 + 1) .$$

Further, we have from (1.1) and (1.3)

$$\text{(3.10)} \qquad \log g_\delta(r, f_1 \dots f_m) = o(1) + \frac{\delta+1}{r^{\delta+1}} \int_{r_0}^{r} \log G(x, f_1 \dots f_m)\, x^\delta\, dx .$$

Substituting for $\log G(x, f_1 \dots f_m)$ from (3.9) in (3.10), we obtain

$$\log g_\delta(r, f_1 \dots f_m) > O(1) + \frac{(\beta - \varepsilon)(\delta + 1)}{\varrho(\varrho + \delta + 1)} \frac{r^{\varrho+\delta+1} - r_0^{\varrho+\delta+1}}{r^{\delta+1}}, \qquad (r_0 \geqslant r_1 + 1).$$

Dividing this throughout by r^ϱ and proceeding to limits, the result follows.

On the other hand, we have from (3.1) for any $\varepsilon > 0$ and $r > r_2 = r_2(\varepsilon)$,

$$n(r, f_1) + \dots + n(r, f_m) < (\alpha + \varepsilon) r^\varrho.$$

Substituting this in (3.8), we get

$$\text{(3.11)} \qquad \log G(r, f_1 \dots f_m) < \log G(r_1, f_1 \dots f_m) + (\alpha + \varepsilon) \frac{r^\varrho - r_1^\varrho}{\varrho}, \qquad (r_1 \geqslant r_2 + 1).$$

Now, substituting this in (3.10), we obtain

$$\log g_\delta(r, f_1 \dots f_m) < O(1) + \frac{(\alpha + \varepsilon)(\delta + 1)}{\varrho(\varrho + \delta + 1)} \frac{r^{\varrho+\delta+1} - r_0^{\varrho+\delta+1}}{r^{\delta+1}}, \qquad (r_0 \geqslant r_1 + 1).$$

Dividing this by r^ϱ and proceeding to limits, the result follows.

Corollary. *We have*

$$\frac{\beta}{\varrho} \leqslant \liminf_{r \to \infty} \frac{\log G(r, f_1 \dots f_m)}{r^\varrho} \leqslant \limsup_{r \to \infty} \frac{\log G(r, f_1 \dots f_m)}{r^\varrho} \leqslant \frac{\alpha}{\varrho}.$$

These easily follow from (3.9) and (3.11) respectively.

4. – Theorem 5. *If $f_1(z), \dots, f_m(z)$ are m integral functions of finite orders $\varrho_1, \dots, \varrho_m$ respectively and if*

$$\text{(4.1)} \qquad \lim_{r \to \infty} \frac{\log G(r, f_1 \dots f_m)}{r^\varrho} = l, \qquad \varrho = \max(\varrho_1, \dots, \varrho_m)$$

exists, then

$$\text{(4.2)} \qquad \lim_{r \to \infty} \frac{n(r, f_1) + \dots + n(r, f_m)}{r^\varrho} = \varrho l,$$

where $f_s(0) \neq 0$ $(s = 1, 2, \dots, m)$.

Proof. We have from (4.1)

$$r^{\varrho}(l-\varepsilon) < \log G(r, f_1 \dots f_m) < r^{\varrho}(l+\varepsilon),$$

for $r > r_0 = r_0(\varepsilon)$ and $\varepsilon > 0$.

Also, for $0 < \eta < 1$

$$\int_r^{(1+\eta)r} \{n(x, f_1) + \dots + n(x, f_m)\} \frac{dx}{x} =$$

$$= \left(\int_0^{(1+\eta)r} - \int_0^r \right) \{n(x, f_1) + \dots + n(x, f_m)\} \frac{dx}{x}$$

$$= \log G(\overline{1+\eta}\, r, f_1 \dots f_m) - \log G(r, f_1 \dots f_m)$$

$$< (l+\varepsilon)(1+\eta)^{\varrho} r^{\varrho} - (l-\varepsilon) r^{\varrho} = l(\varrho\eta + \dots) r^{\varrho} + \varepsilon(2 + \varrho\eta + \dots) r^{\varrho},$$

but,

$$\int_r^{(1+\eta)r} \{n(x, f_1) + \dots + n(x, f_m)\} \frac{dx}{x} \geqslant \{n(r, f_1) + \dots + n(r, f_m)\} \int_r^{(1+\eta)r} \frac{dx}{x}$$

$$> \{n(r, f_1) + \dots + n(r, f_m)\} \frac{\eta}{1+\eta},$$

giving

$$\frac{n(r, f_1) + \dots + n(r, f_m)}{r^{\varrho}} < \frac{l(1+\eta)(\varrho\eta + \dots)}{\eta} + \frac{\varepsilon(2 + \varrho\eta + \dots)(1+\eta)}{\eta}.$$

Since ε and η are arbitrary, this gives

$$\limsup_{r\to\infty} \frac{n(r, f_1) + \dots + n(r, f_m)}{r^{\varrho}} \leqslant \varrho\, l. \tag{4.3}$$

Further, it can easily be shown that

$$\frac{n(r, f_1) + \dots + n(r, f_m)}{r^{\varrho}} > \frac{l(1-\eta)(\varrho\eta - \dots)}{\eta} - \frac{\varepsilon(2 - \varrho\eta + \dots)(1-\eta)}{\eta},$$

which leads to

$$\liminf_{r\to\infty} \frac{n(r, f_1) + \dots + n(r, f_m)}{r^{\varrho}} \geqslant \varrho\, l \,. \tag{4.4}$$

This completes the proof of (4.2) .

Theorem 6. *Let $f_1(z), \dots, f_m(z)$ be m integral functions, other than polynomials, of orders $\varrho_1, \dots, \varrho_m$ respectively and let $n(r, f_1), \dots, n(r, f_m)$ denote the zeros of $f_1(z), \dots, f_m(z)$ respectively in $|z| \leqslant r$ and $f_s(0) \neq 0$ $(s = 1, 2, \dots, m)$. Further, if*

$$\liminf_{r\to\infty} \frac{n(r, f_1) + \dots + n(r, f_m)}{r \log r} > 1 \,, \tag{4.5}$$

then

$$\liminf_{r\to\infty} \frac{\log g_\delta(r, f_1 \dots f_m)}{r \log r} > \frac{\delta + 1}{\delta + 2} \,, \tag{4.6}$$

and if

$$\limsup_{r\to\infty} \frac{n(r, f_1) + \dots + n(r, f_m)}{r \log r} < 1 \,, \tag{4.7}$$

then

$$\limsup_{r\to\infty} \frac{\log g_\delta(r, f_1 \dots f_m)}{r \log r} < \frac{\delta + 1}{\delta + 2} \,. \tag{4.8}$$

Proof. From (4.5), we have for any $\varepsilon > 0$ and $r > r_2 = r_2(\varepsilon)$

$$n(r, f_1) + \dots + n(r, f_m) > (1 - \varepsilon)\, r \log r \,.$$

Substituting this in

$$\left\{ \begin{aligned} &\log G(r, f_1 \dots f_m) = \\ &= \log G(r_1, f_1 \dots f_m) + \int_{r_1}^{r} \{n(x, f_1) + \dots + n(x, f_m)\} \frac{dx}{x} \,, \end{aligned} \right. \tag{4.9}$$

we obtain

$$\text{(4.10)} \qquad \log G(r, f_1 \dots f_m) > \text{const.} + (1-\varepsilon)\, r\, (\log r - 1) \qquad (r_1 \geqslant r_2 + 1)\,.$$

Substituting for $\log G(x, f_1 \dots f_m)$ from (4.10) in

$$\text{(4.11)} \qquad \log g_\delta(r, f_1 \dots f_m) = o(1) + \frac{\delta+1}{r^{\delta+1}} \int_{r_0}^{r} \log G(x, f_1 \dots f_m)\, x^\delta \, dx\,,$$

where $r_0 \geqslant r_1 + 1$, we obtain

$$\log g_\delta(r, f_1 \dots f_m) > (1-\varepsilon)\, \frac{\delta+1}{\delta+2} \cdot r \log r\,.$$

Dividing this by $r \log r$ and proceeding to limits, (4.6) follows.

On the other hand, for any $\varepsilon > 0$ and $r > r_2 = r_2(\varepsilon)$, we have from (4.7)

$$n(r, f_1) + \dots + n(r, f_m) < (1+\varepsilon)\, r \log r\,,$$

which together with (4.9) gives

$$\text{(4.12)} \qquad \log G(r, f_1 \dots f_m) < \text{const.} + (1+\varepsilon)\, r (\log r - 1) \qquad (r_1 \geqslant r_2 + 1)\,.$$

Substituting this in (4.11), we have

$$\log g_\delta(r, f_1 \dots f_m) < (1+\varepsilon)\, \frac{\delta+1}{\delta+2}\, r \log r\,,$$

from which (4.8) follows immediately.

Corollary. *We have*

$$\liminf_{r\to\infty} \frac{\log G(r, f_1 \dots f_m)}{r \log r} > 1\,,$$

provided (4.5) *holds, and*

$$\limsup_{r\to\infty} \frac{\log G(r, f_1 \dots f_m)}{r \log r} < 1\,,$$

provided (4.7) *holds.*

These results are immediate consequences of (4.10) and (4.12) respectively.

5. – Theorem 7. *For m integral functions of finite orders $\varrho_1, \ldots, \varrho_m$, respectively,*

$$G(r, f_1^{(1)} \ldots f_m^{(1)}) < K\, G(r, f_1 \ldots f_m)\, r^{\varrho_1 + \ldots + \varrho_m - m + \varepsilon}, \tag{5.1}$$

except at a set of measure zero, for every $\varepsilon > 0$ and large r, where constant K is independent of r.

Proof. We have

$$G(r, f_1^{(1)} \ldots f_m^{(1)}) =$$

$$= \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \log |f_1^{(1)}(r e^{i\theta}) \ldots f_m^{(1)}(r e^{i\theta})|\, d\theta\right\} =$$

$$= G(r, f_1 \ldots f_m)\ \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \log\left|\frac{f_1^{(1)}(r e^{i\theta})}{f_1(r e^{i\theta})} \cdots \frac{f_m^{(1)}(r e^{i\theta})}{f_m(r e^{i\theta})}\right| d\theta\right\}.$$

But, we know ([2], p. 363) that

$$\left|\frac{f^{(1)}(r e^{i\theta})}{f(r e^{i\theta})}\right| \leqslant O(r^{\varrho - 1 + \varepsilon}),$$

for every $\varepsilon > 0$ and large r outside a set of measure zero. Using this for the functions $f_1, \ldots, f_m$ in the above result, we obtain

$$G(r, f_1^{(1)} \ldots f_m^{(1)}) < K\, G(r, f_1 \ldots f_m)\, r^{\varrho_1 + \ldots + \varrho_m - m + \varepsilon}.$$

Corollary. 1. *We have*

$$\limsup_{r \to \infty} \left\{\log\left(r^m \frac{G(r, f_1^{(1)} \ldots f_m^{(1)})}{G(r, f_1 \ldots f_m)}\right) \Big/ \log r\right\} \leqslant \varrho_1 + \ldots + \varrho_m .$$

Corollary 2. *For integral functions $f_1(z), \ldots, f_m(z)$ of finite orders $\varrho_1, \ldots, \varrho_m$ respectively*

$$\limsup_{r \to \infty} \left[\log\left(r^m \left\{\frac{G(r, f_1^{(n)} \ldots f_m^{(n)})}{G(r, f_1 \ldots f_m)}\right\}^{1/n}\right) \Big/ \log r\right] \leqslant \varrho_1 + \ldots + \varrho_m .$$

Writing (5.1) for the s-th derivatives of $f_1(z), \ldots, f_m(z)$, we have

$$\frac{G(r, f_1^{(s)} \ldots f_m^{(s)})}{G(r, f_1^{(s-1)} \ldots f_m^{(s-1)})} < K_s\, r^{\varrho_1+\ldots+\varrho_m-m+\varepsilon} .$$

Giving s the values $s = 1, 2, \ldots, n$, multiplying all the inequalities thus obtained, replacing $K_1, \ldots, K_n$ by K, where $K = \max(K_1, \ldots, K_n)$ and proceeding to limits the result follows.

Theorem 8. *If $f_1(z), \ldots, f_m(z)$ are m integral functions of finite orders $\varrho_1, \ldots, \varrho_m$ respectively, then*

$$\limsup_{r\to\infty} \left\{\log\left(r^m \frac{g_\delta(r, f_1^{(1)} \ldots f_m^{(1)})}{g_\delta(r, f_1 \ldots f_m)}\right) \Big/ \log r\right\} \leqslant \varrho_1 + \ldots + \varrho_m , \tag{5.2}$$

where r tends to infinity through values outside a set of measure zero.

Proof. We have

$$\log g_\delta(r, f_1^{(1)} \ldots f_m^{(1)}) = \frac{\delta+1}{r^{\delta+1}} \int\limits_0^r \log G(x, f_1^{(1)} \ldots f_m^{(1)})\, x^\delta \,\mathrm{d}x$$

$$= o(1) + \frac{\delta+1}{r^{\delta+1}} \int\limits_{r_0}^r \log G(x, f_1^{(1)} \ldots f_m^{(1)})\, x^\delta \,\mathrm{d}x .$$

Using (5.1) in this, we obtain

$$\log g_\delta(r, f_1^{(1)} \ldots f_m^{(1)}) \leqslant$$

$$\leqslant O(1) + \log g_\delta(r, f_1 \ldots f_m) + \frac{\delta+1}{r^{\delta+1}} \int\limits_{r_0}^r (\varrho_1 + \ldots + \varrho_m - m + \varepsilon) \log x \cdot x^\delta \,\mathrm{d}x =$$

$$= O(1) + \log g_\delta(r, f_1 \ldots f_m) + (\varrho_1 + \ldots + \varrho_m - m + \varepsilon) \log r ,$$

or

$$\log\left(r^m \frac{g_\delta(r, f_1^{(1)} \ldots f_m^{(1)})}{g_\delta(r, f_1 \ldots f_m)}\right) \leqslant O(1) + (\varrho_1 + \ldots + \varrho_m + \varepsilon) \log r .$$

Proceeding to limits in this, (5.2) follows.

Corollary. *We have*

$$\limsup_{r\to\infty}\left[\log\left(r^m\left\{\frac{g_\delta(r, f_1^{(n)}\dots f_m^{(n)})}{g_\delta(r, f_1\dots f_m)}\right\}^{1/n}\right)\Big/ \log r\right] \leqslant \varrho_1 + \dots + \varrho_m\,.$$

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