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An Extension of a Class of Polynomials (I). (**)

(Dedicated to Prof. L. CARLITZ on his 60^{th} birthday.)

1. - Introduction.

More than two centuries have passed since EULER gave the identity

$$(x+y)(x+qy)\dots(x+q^{n-1}y)=\sum_{r=0}^{n}[n,r]\,q^{r(r-1)/2}\,x^{n-r}\,y^{r},$$

where:

$$[n,r]=\frac{[n]!}{[r]!\,[n-r]!},$$

$$[n]=(q^{n}-1)/(q-1)\,, \qquad q \text{ is a fixed complex number, } |q|\neq 1\,,$$

$$[n]!=[n][n-1]\dots[1]\,, \qquad [1]!=[0]!=1\,.$$

Basing his study essentially on this famous identity, F. H. JACKSON in a series of papers written in the first-half of the present century developed the q-difference calculus. In 1936 MORGAN WARD [9] wrote a remarkable paper generalizing a large portion of algebraic analysis and the calculus of finite differences including a great variety of formulas involving the exponential function and the BERNOULLI numbers and polynomials. He starts with a given

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sequence (u) of complex numbers subject to the very few and simple restrictions:

$$u_0 = 0 , \qquad u_1 = 1 , \qquad u_n \neq 0 \quad \text{for} \quad n > 1 ;$$

and develops a calculus of sequences. He, however, does not go deep enough to bring out the rather obvious fact that one would get as a particular case the q-difference calculus of JACKSON, i.e., for which $u_n = (q^n - 1)/(q - 1)$, if and only if the law of exponents $u^n u^m = u^{n+m}$ holds true, where $u^n \equiv u_{n+1} - u_n$. A host mathematicians during the present century have contributed to the study of this rather restricted part of the calculus of sequences and numerous papers are still being written on it.

2. - Object of present study.

A system of polynomials $\{P_n(x)\}$ satisfying the property

$$\text{(1)} \qquad \mathrm{D} P_n(x) = P_{n-1}(x) \qquad (n = 0, 1, 2, \dots)$$

is called an APPELL set and has received considerable attention since its introduction in 1880 by P. APPELL [2]: here D is the ordinary differential operator, i.e., $\mathrm{D} f(x) = \operatorname*{Lt}_{h \to 0} \{f(x + h) - f(x)\}/h$. It was in 1954 that SHARMA and CHAK [6] studied a class of polynomials $\{H_n(x)\}$ such that

$$\text{(2)} \qquad \mathrm{D}_q H_n(x) = H_{n-1}(x) \qquad (n = 0, 1, 2, \dots) ,$$

where D_q is the q-difference operator of JACKSON and is given by

$$\mathrm{D}_q f(x) = \{f(qx) - f(x)\}/\{(q - 1)x\} .$$

Very recently AL-SALAM [1] has discussed in detail the algebraic structure and some other properties of the class of polynomials satisfying (2); he calls these the « q-APPELL polynomials ».

In this paper we take up the general calculus of finite differences in which the law of exponents for $u^n \equiv u_{n+1} - u_n$ does not necessarily hold and study the polynomial systems $\{H_n(x)\}$ in x, such that

$$\text{(3)} \qquad \mathrm{D}_u H_n(x) = H_{n-1}(x) \qquad (n = 0, 1, 2, \dots) .$$

Here D_u is a very general linear distributive operator (see WARD [9]) which is such that it converts a polynomial of degree n in x into one of degree $n - 1$; more specifically,

$$\mathrm{D}_u x^n = u_n \, x^{n-1} ,$$

where $\{u_n\}$ is the given sequence of complex numbers used by WARD [9] to develop his unrestricted calculus.

In the next paper we will examine some subsets of this class of polynomials satisfying (3) wich have properties analogous to the regular and cyclic sets of NIELSEN [5] and WARD [8] and study the algebraic structure of this set on the lines of AL-SALAM [1] analogous to those of the « q-APPELL sets »; we will also give a short history and reference to recent work on the subject.

3. - Preliminaries.

In the terminology of WARD [9] let

$$(u): \qquad u_0 = 0\,, \quad u_1 = 1\,, \quad u_2\,, \quad \dots\,, \quad u_n\,, \quad \dots$$

be a fixed sequence of real or complex numbers subject to the single restriction $u_n \neq 0$ for $n = 2, 3, 4, \dots$. We will sometimes use $[n]$ for u_n, defining

$$[n]! = [n][n-1] \dots [1] \text{ if } n > 0\,, \qquad [n]! = 1 \text{ if } n = 0\,,$$

$$[n, r] = \frac{[n][n-1] \dots [n-r+1]}{[r]!} = \frac{[n]!}{[r]!\,[n-r]!}\,,$$

where n and r are positive integers and $n \geqslant r$; thus

$$[n, 1] = [n]\,, \qquad [n, 0] = 1\,, \qquad [n, n-r] = [n, r]\,,$$

also $[n, r] = 0$ if $n < r$ or $r < 0$. We shall call $[n, r]$ a binomial coefficient to the base (u).

We next define an operator D_u which transforms the formal power series

$$F(x) = \sum_{n=0}^{\infty} c_n x^n \quad \text{into} \quad F_u'(x) = \mathrm{D}_u F(x) = \sum_{n=0}^{\infty} [n] c_n x^{n-1}\,.$$

In particular $\mathrm{D}_u x^n = u_n x^{n-1}$. If we define

$$F_u^{(r+1)}(x) = \mathrm{D}_u F^{(r)}(x)\,, \quad \text{where} \quad F_r^{(r)}(x) = \mathrm{D}_u^r F(x) \quad \text{and} \quad F_u^{(0)}(x) = F(x)\,,$$

we have
$$F_u^{(r)}(x)/[r]! = \sum_{n=r}^{\infty} [n, r] c_n x^{n-r}\,.$$

If $H_n(x) = \sum_{i=0}^{n} a_{n,i}\, x^{n-i}$, then we define

$$H_n[ax+b] = \sum_{i=0}^{n} a_{n,i} [ax+b]_{n-i}\,,$$

where

$$[ax+b]_n = (ax+b)(ax+bu^1)(ax+bu^2)\dots(ax+bu^{n-1}) = \sum_{r=0}^{n} {}^nS_r(u)\,(ax)^{n-r}b^r\,;$$

here ${}^nS_r(u)$ means the sum of all the combinations of the products of n numbers $u^0, u^1, u^2, \dots, u^{n-1}$ taken r at a time and $u^n \equiv u_{n+1} - u_n$. It is interesting to note that

$${}^nS_{n-r}(u) \neq {}^nS_r(u)$$

though the number of terms is the same in both viz. $n!/\{r!\,(n-r)!\}$. We also notice that

$$\sum_{r=0}^{n} (-1)^r\,{}^nS_r(u) = \begin{cases} 1 & \text{if } n=0 \\ 0 & \text{if } n \geqslant 1\,. \end{cases}$$

In passing we give a short proof of the important fact that if $u^n \equiv u_{n+1} - u_n$ and the law of exponents holds, i.e. if

$$u^m\,u^n = u^{m+n} \qquad (m, n = 0, 1, 2, \dots)\,,$$

then the sequence (u) is actually given by $u_n = \{(u^1)^n - 1\}/(u^1 - 1)$ and this corresponds exactly to the case of JACKSON's q-numbers; as $q \to 1$ the latter sequence gives the set of non-negative integers. For proof of this it is sufficient if we just concentrate our attention on the following identity:

$$u_n = u^0 + u^1 + u^2 + \dots + u^{n-1}\,.$$

Finally we note that $H_n[b+x]$ is not the same as $H_n[x+b]$ and that

$$H_n[x] = \sum_{i=0}^{n} a_{n,i}\Big(\prod_{r=0}^{n-i-1} u^r\Big)\,x^{n-i} = \sum_{i=0}^{n} a_{n,i}\,{}^{n-i}S_{n-i}(u)\,x^{n-i}\,.$$

4. - Appell polynomial sets (or harmonic sequences) to the base (u).

In analogy with NIELSEN [5] we shall call the polynomial sets $\{H_n(x)\} \equiv \{H_n(x, u)\}$ which satisfy the functional relation (3) of § **2** to be « harmonic sequences to the base (u) » (see also [4], [6]); following WARD [**9**] and AL-SALAM [**1**] we may call them « APPELL polynomial sets to the base (u) »; they can also be called « generalized BRENKE polynomials » (see [3], pp. 44-45).

A simple example of such a set is $\{H_n(x)\}$, where $H_n(x) = \sum_{k=0}^{n} x^k/\{u_{n-k}!\;u_k!\}$.

We first give three characterizations of our polynomial sets in the form of theorems.

Theorem I. *If* $\{H_n(x)\}$ *is a harmonic sequence to the base* (u), *then there exists a sequence of constants* $\{h_n\}$ *such that*

$$H_n(x) = h_0 \frac{x^n}{u_n!} + h_1 \frac{x^{n-1}}{u_{n-1}!} + \dots + h_n\,, \qquad H_n(0) = h_n\,.$$

We will sometimes indicate the two by $[H_n(x), h_n]$.

For the proof we observe that $a_{n,i}\, u_{n-i} = a_{n-i,i}$. Sufficiency of the condition is obvious.

Theorem II. *If* $[H_n(x), h_n]$ *is a harmonic sequence to the base* (u), *then*

$$\sum_{n=0}^{\infty} H_n(x)\, t^n = e_u(x\,t)\, h(t)\,,$$

where

$$e_u(x) = \sum_{n=0}^{\infty} x^n / u_n! \qquad \textit{and} \qquad h(t) = \sum_{n=0}^{\infty} h_n t^n\,.$$

We say $h(t)$ is the determining function of the polynomial set $H_n(x)$.

For the proof it is easy to see that

$$\sum_{n=0}^{\infty} H_n(x)\, t^n = \sum_{n=0}^{\infty} t^n \sum_{r=0}^{n} h_r \frac{x^{n-r}}{u_{n-r}!} = \sum_{n=0}^{\infty} \frac{(x\,t)^n}{u_n!} \sum_{n=0}^{\infty} h_n t^n\,.$$

Sufficiency of the condition is also easy to prove.

It is assumed here that the sequence (u) is so chosen that the « general exponential series $e_u(x)$ » is convergent in the neighborhood of $x = 0$. It is accordingly an element of an analytic function of x and may be called the « (u)-basic exponential ». There exists then a positive number ϱ such that this series converges absolutely within the circle $|x| = \varrho$ (see [9] and also [3]).

Theorem III. *The necessary and sufficient condition that* $\{H_n(x)\}$ *be a harmonic sequence to the base* (u) *is that there exists a function of bounded variation* $\beta(x) \equiv \beta(x, u)$ *on* $(0, \infty)$ *so that, for* $n = 0, 1, 2, \dots$,

(i) $b_n = \int_0^{\infty} x^n \, d\beta(x)$ *exist*, $b_0 \neq 0$;

(ii) $H_n(x) = (1/u_n!) \int_0^{\infty} [x + t]^n \, d\beta(t)$, *where* $[x + t]^n = \sum_{k=0}^{n} [n, k]\, x^{n-k} t^k$.

Following SHEFFER [7] we observe that if (i) holds, then $H_n(x)$ as given by (ii) exists for each n, and is the polynomial of degree exactly n. Moreover, since it is permissible to interchange the order of integration and differencing $D_u H_n(x) = H_{n-1}(x)$; that is, $\{H_n(x)\}$ is a harmonic sequence to the base (u).

Now suppose $\{H_n(x)\}$ is a harmonic sequence to the base (u), then following the lines of SHEFFER [7] it is easy to show that its determining function $A(t)$ is given by

$$A(t) = \int_0^\infty e_u(x\,t)\,d\beta(x) = \sum_{n=0}^\infty (b_n/u_n!)\,t^n .$$

We now give some other properties of our polynomial sets.

Theorem IV. *If $[H_n(x), h_n]$ is a harmonic sequence to the base (u), then we have:*

(a) $$D_u H_n[x] = H_{n-1}[u\,x] ,$$

(b) $$H_n[x+b] = \sum_{i=0}^{n} (x^i/u_i!)\,H_{n,i}[b] ,$$

(c) $$H_n[b+x] = \sum_{i=0}^{n} x^i H_{n,i}(b) ;$$

here

$$H_{n,i}[x] = \sum_{r=0}^{n-1} a_{n-i,r}\,u_{n-i-r}!\,\alpha_{n-r,\,n-i-r}\,x^{n-i-r} , \qquad H_{n,i}(x) = \sum_{r=0}^{n-i} a_{n-i,r}\,\alpha_{n-r,i}\,x^{n-i-r} ,$$

$$^{n}S_r(u) = (u_n!/u_{n-r}!)\,\alpha_{n,r}$$

and $\alpha_{n,r}$ satisfies the recurrence relations:

$$\begin{cases} \alpha_{n,0} = 1 \quad (n = 0, 1, 2 \ldots), \qquad \alpha_{n,n} = u^0 u^1 \ldots u^{n-1}/u_n! \quad (n = 1, 2, \ldots) \\ u_n \alpha_{n,r} = u_{n-r}\alpha_{n-1,r} + u^{n-1}\alpha_{n-1,r-1} \quad (n = 1, 2, \ldots;\ r = 0, 1, 2, \ldots, n) . \end{cases}$$

We notice that $H_{n,0}[x] = H_n[x]$ and $H_{n,0}(x) = H_n(x)$.

Theorem V. *If $[H_n(x), h_n]$ and $[K_n(x), k_n]$ are two harmonic sequences to the base (u), then the expression*

$$A_n(x) = \sum_{s=0}^{n} (-1)^s H_{n-s}(x)\,K_s\{x\}$$

is a constant, where

$$K_n\{x\} = \sum_{i=0}^{n} d_i k_{n-i} x^i , \qquad \sum_{s=0}^{n} (-1)^s d_s = 0 , \qquad d_0 = 1 .$$

It is easy to see that $A_n(x) = \sum_{s=0}^{n} h_{n-s} k_s$.

Theorem VI. *If $[H_n(x), h_n]$ and $[K_n(x), k_n]$ are two harmonic sequences to the base (u), then*

(a) *there exists a unique sequence $\{\alpha_n\}$ such that, for all n,*

$$K_n(x) = \alpha_0 H_n(x) + \alpha_1 H_{n-1}(x) + \ldots + \alpha_n H_0(x) ;$$

(b) *if only one of the two harmonic sequences is given, the second is completely determined provided that either of the two following functional relationship holds:*

(i) $$H_n[x] - H_n[-1 + x] = K_{n-1}(x) ,$$

(ii) $$H_n[x] + H_n[-1 + x] = K_n(x) .$$

For the proof of (a) we notice that $\alpha_0, \alpha_1, \ldots, \alpha_n$ can be uniquely determined from $n + 1$ equations got by equating like powers of x in

$$\sum_{i=0}^{n} \frac{k_i}{u_{n-i}!} x^{n-i} = \alpha_0 \sum_{i=0}^{n} \frac{h_i}{u_{n-i}!} x^{n-i} + \alpha_1 \sum_{i=0}^{n-1} \frac{h_i}{u_{n-i-1}!} x^{n-i-1} + \ldots + \alpha_n h_0 .$$

We leave the proof of (b) to the reader.

5. -

Various interesting examples can be found of general sequences (u) for wich the law of exponents $u^m u^n = u^{m+n}$ does not hold for $u^n \equiv u_{n+1} - u_n$ and of the corresponding linear distributive operator D_u which transforms a polynomial of degree n in x into one of degree $n - 1$ but the most interesting seems to be the following simple concrete example given by Ward [9]. Assuming that a sequence (u) is a linear recurring series of order k whose associated polynomial $x^k - a_1 x^{k-1} - \ldots - a_k$ has k distinct roots $\alpha_1, \alpha_2, \ldots, \alpha_k$, then u_n is given by

$$u_n = \beta_1 \alpha_1^n + \beta_2 \alpha_2^n + \ldots + \beta_k \alpha_k^n ,$$

where α' s and β' s are constants subject to the conditions

$$\beta_1 + \beta_2 + \ldots + \beta_k = 0\,, \qquad \alpha_1\beta_1 + \alpha_2\beta_2 + \ldots + \alpha_k\beta_k = 1\,,$$

and $u_n \neq 0$ for $n > 1$. Now $\mathrm{D}_u F(x)$ is given by

$$\mathrm{D}_u F(x) = \frac{\beta_1 F(\alpha_1 x) + \beta_2 F(\alpha_2 x) + \ldots + \beta_k F(\alpha_k x)}{(\alpha_1\beta_1 + \alpha_2\beta_2 + \ldots + \alpha_k\beta_k)\, x}\,.$$

This operator can therefore be applied to any function of x regular at $x = 0$ and transforms it into another function regular at $x = 0$.
In particular if

$$k = 2\,, \qquad \alpha_1 = q\,, \qquad \alpha_2 = 1\,, \qquad \beta_1 = 1/(q-1) = -\beta_2\,,$$

where q is not a root of unity, then

$$\mathrm{D}_u F(x) = \{F(qx) - F(x)\}/\{(q-1)\,x\}\,,$$

i.e. $\mathrm{D}_u \equiv \mathrm{D}_q$ the operation of q-differencing. We observe in passing, that as $q \to 1$, $\mathrm{D}_q \to \mathrm{D} \equiv \mathrm{d}/\mathrm{d}x$, the ordinary derivative. In case some of the roots α of the polynomial associated with the recurrence relation are repeated, a similar but more complicated formulae for $\mathrm{D}_u F(x)$ may be given which involve both $F(x)$ and its ordinary derivatives. For example, if $k = 2$, $\alpha_1 = \alpha_2 \neq 0$, then

$$u_n = n\,\alpha_1^{n-1} \qquad \text{and} \qquad \mathrm{D}_u F(x) = \frac{1}{\alpha_1}\,\frac{\mathrm{d}}{\mathrm{d}x}\,F(\alpha_1 x).$$

For the sake of illustration of what these theorems say let us just pick up Theorem IV (a). In the light of the above example it reads as follows:

If $H_n(x) = \sum_{i=0}^{n} a_{n,i}\, x^{n-i}$ is a harmonic sequence to the base (u), then

$$\mathrm{D}_u \sum_{i=0}^{n} a_i{}^{n-i} S_{n-i}(u)\, x^{n-i} = \sum_{i=0}^{n-1} a_i{}^{n-i-1} S_{n-i-1}(u)\,(ux)^{n-i-1}\,,$$

where

$$\mathrm{D}_u F(x) = \frac{1}{x} \sum_{s=1}^{k} \beta_s F(\alpha_s x) \qquad \text{and} \qquad u^r = \sum_{s=1}^{k} \beta_s\,(\alpha_s^{r+1} - \alpha_s^r)\,.$$

If $k = 2$, $\alpha_1 = 1$, $\alpha_2 = q$, then $u_n = (q^n - 1)/(q-1)$, $u^r = q^r$ and we can use the law of exponents to get

$$D_q \sum_{i=0}^{n} a_i\, q^{(n-i)(n-i-1)/2}\, x^{n-i} = \sum_{i=0}^{n-1} a_i\, q^{(n-i-1)(n-i-2)/2}\, (qx)^{n-i-1}$$

which incidently is property (ii) of SHARMA and CHAK [6]; if further $q \to 1$ it gives us the simple APPELL property, i.e. the polynomial set satisfies the functional relationship (1) of § 2 of the present paper.

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