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SHIVA NARAIN LAL (*)

On the Absolute Nörlund Summability of Fourier Series. (**)

1.1. – Let $\sum a_n$ be a series with partial sums s_n and let $\{p_n\}$ be a sequence of real constants with

$$P_n = p_0 + p_1 + p_2 + \ldots + p_n \qquad (P_{-1} = p_{-1} = 0).$$

The series $\sum a_n$ is said to be summable $|N, p_n|$ if

$$\sum_{n=1}^{\infty} | t_n - t_{n-1} | < \infty ,$$

where

$$t_n = (1/P_n) \sum_{\nu=0}^{n} p_{n-\nu} s_\nu .$$

If we take $p_n = 1/(n+1)$, the NÖRLUND mean $\{t_n\}$ reduces to the familiar harmonic mean [7].

In the sequel it is assumed that the sequence $\{p_n\}$ is non-negative, non-increasing and $\lim_{n\to\infty} p_n = 0$.

1.2. – Let $f(t)$ be a periodic function with period 2π and integrable (L) in $(-\pi, \pi)$. The FOURIER series of $f(t)$ is

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(*) Indirizzo: Department of Mathematics, Faculty of Science, Banaras Hindu University, Varanasi-5, India.

(**) Ricevuto: 1-X-1969.

$$\tfrac{1}{2} a_0 + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=0}^{\infty} A_n(t) ,$$

where a_n and b_n are given by the usual EULER-FOURIER formulae. We write

$$\varphi(t) = f(x + t) + f(x - t) - 2 f(x);$$

$$\alpha(t) = \sum_{\nu=0}^{\infty} p_\nu \cos \nu t , \qquad \beta(t) = \sum_{\nu=0}^{\infty} p_\nu \sin \nu t ;$$

$$\alpha_n = \int_0^{\pi} \varphi(t)\, \alpha(t) \cos nt \, dt , \qquad \beta_n = \int_0^{\pi} \varphi(t)\, \beta(t) \sin nt \, dt ;$$

x denotes a variable increasing to infinity and t tends to zero from the right.

2.1. – In this paper, which is in continuation of a series of papers [1], [2], [3], [4] devoted to the study of the absolute NÖRLUND summability of infinite series, we establish the following

Theorem. *Let the sequence* $\{p_n - p_{n+1}\}$ *be non-increasing and*

(i) $\sum_{n=1}^{\infty} P_n^2 n^{-2} < C$ (¹). *Let* χ *be a function which increases to infinity and which is such that*

(ii) $P(y)/\chi(y)$ *is decreasing,*

(iii) $t^{\varepsilon} \chi(t^{-1})$ *increases as* t *increases* ($\varepsilon > 0$, *and small*) *and further,*

(iv) $\sum_{n=1}^{\infty} \frac{1}{n\,\chi(n)} < C,$

(v) $\sum_{n=1}^{\infty} \frac{1}{n\,\chi^{1/2}(n)\, P_n} < C .$ *If* $f(x)$ *is of bounded variation and*

(vi) $|\varphi(t)|\, \chi(t^{-1}) = O(1),$

then the series $\sum A_n(x)$ *is summable* $|N, p_n|$.

(¹) Throughout the paper C denotes an absolute constant not necessarily the same at each occurrence.

2.2. – The following lemmas will be required in the proof of the theorem.

Lemma 1 ([5], Lemma 5.11). For $0 \leqslant a < b \leqslant \infty$, $0 < t \leqslant \pi$ and any n:

$$\left| \sum_{k=a}^{b} p_k e^{i(n-k)t} \right| \leqslant C P_\tau \qquad (\tau = [t^{-1}]).$$

Lemma 2 ([5], Lemma 5.20). For t in (h, π):

$$| \gamma(t+2h) - \gamma(t) | \leqslant C h t^{-1} P(h^{-1}),$$

where

$$\gamma(t) = \sum_{\nu=0}^{\infty} p_\nu e^{i\nu t}.$$

Lemma 3 ([5], Lemmas 5.12 and 5.14). For $t \leqslant 1/n$:

$$p_n \sum_{\nu=0}^{n} P_\nu / P_n \leqslant C P(t^{-1}), \tag{2.2.1}$$

and

$$\sum_{n=k}^{\infty} \frac{n \, \Delta p_n}{P_n P_{n-1}} \leqslant C P_k^{-1}. \tag{2.2.2}$$

Lemma 4 ([5], see proof of Lemma 5.16). If $\{p_n - p_{n+1}\}$ is non-increasing, then

$$\frac{1}{P_{n-1}} \left| \int_{1/n}^{\pi} \varphi(t) \left\{ \sum_{k=n}^{\infty} p_k \cos(n-k)t + \sum_{k=0}^{n-1} \frac{p_n}{P_n} P_k \cos(n-k)t \right\} dt \right| \leqslant$$

$$\leqslant C \left(\frac{n \, \Delta p_n}{P_n P_{n-1}} + \frac{p_n}{P_n P_{n-1}} \right) \int_{1/n}^{\pi} | \varphi(t) | P(t^{-1}) t^{-1} dt + C \frac{p_n^2}{P_n P_{n-1}} \int_{1/n}^{\pi} | \varphi(t) | dt.$$

2.3. - Proof of the Theorem.

We have

$$A_n(x) = (1/\pi) \int_0^{\pi} \varphi(t) \cos nt \, dt$$

and therefore

$$\pi \, | t_n - t_{n-1} | = \frac{1}{P_n P_{n-1}} \left| \int_0^\pi \varphi(t) \sum_{k=0}^{n-1} (p_k P_n - p_n P_k) \cos(n-k)t \, dt \right| \leqslant$$

$$\leqslant \frac{1}{P_{n-1}} \left| \int_0^\pi \varphi(t) \sum_{k=0}^{\infty} p_k \cos(n-k)t \cdot dt \right| + \frac{1}{P_{n-1}} \left| \int_0^{1/n} \varphi(t) \sum_{k=n}^{\infty} p_k \cos(n-k)t \cdot dt \right| +$$

$$+ \frac{p_n}{P_n P_{n-1}} \left| \int_0^{1/n} \varphi(t) \sum_{k=0}^{n-1} P_k \cos(n-k)t \cdot dt \right| +$$

$$+ \frac{1}{P_{n-1}} \left| \int_{1/n}^{\pi} \varphi(t) \left\{ \sum_{k=n}^{\infty} p_k \cos(n-k)t + \sum_{k=0}^{n-1} \frac{p_n}{P_n} P_k \cos(n-k)t \right\} dt \right|$$

$$= \sum_{r=1}^{4} | \gamma_{n,r} | , \quad \text{say.}$$

Hence, for establishing the theorem, we have to prove that under the hypotheses of the theorem

$$\text{(2.3.1)} \qquad \sum_{n=2}^{\infty} | \gamma_{n,r} | < \infty \qquad (r = 1, 2, 3, 4) .$$

Making use of Lemma 1 and the hypotheses (vi), (ii) and (iv) respectively of the theorem, we get

$$\text{(2.3.2)} \qquad \begin{cases} \displaystyle\sum_{n=2}^{\infty} | \gamma_{n,2} | \leqslant C \sum_{n=2}^{\infty} \frac{1}{P_{n-1}} \int_0^{1/n} | \varphi(t) | \, P(t^{-1}) \, dt \leqslant \\ \displaystyle\leqslant C \sum_{n=2}^{\infty} \frac{1}{P_{n-1}} \int_0^{1/n} \frac{P(t^{-1})}{\chi(t^{-1})} \, dt \leqslant C \sum_{n=2}^{\infty} \frac{1}{n \, \chi(n)} \leqslant C . \end{cases}$$

Also, making use of (2.2.1) of Lemma 3, we get

$$\text{(2.3.3)} \qquad \begin{cases} \displaystyle\sum_{n=2}^{\infty} | \gamma_{n,3} | \leqslant C \sum_{n=2}^{\infty} \frac{1}{P_{n-1}} \int_0^{1/n} | \varphi(t) | \, \frac{P_0 + P_1 + \dots + P_n}{P_n} \, p_n \, dt \\ \displaystyle\qquad \leqslant C \sum_{n=2}^{\infty} \frac{1}{P_{n-1}} \int_0^{1/n} \frac{P(t^{-1})}{\chi(t^{-1})} \, dt \leqslant C , \end{cases}$$

as in the estimate (2.3.2).

Again, making use of Lemma 4 and the hypothesis (vi) of the Theorem, we get

$$(2.3.4)\quad \begin{cases} \displaystyle\sum_{n=2}^{\infty} |\gamma_{n,4}| \leqslant \\ \displaystyle\leqslant C \sum_{n=1}^{\infty} \left(\frac{n\,\Delta p_n}{P_n\,P_{n-1}} + \frac{p_n}{P_n\,P_{n-1}}\right) \int_{1/n}^{\pi} \frac{P(t^{-1})}{t\,\chi(t^{-1})}\,dt + C \sum_{n=1}^{\infty} \frac{p_n^2}{P_n\,P_{n-1}} \\ \displaystyle\leqslant C \sum_{n=1}^{\infty} \left(\frac{n\,\Delta p_n}{P_n\,P_{n-1}} + \frac{p_n}{P_n\,P_{n-1}}\right)\left(1 + \sum_{r=1}^{n} \frac{P_r}{r\,\chi(r)}\right) + C \\ \displaystyle\leqslant C + C \sum_{r=1}^{\infty} \frac{P_r}{r\,\chi(r)} \sum_{n=r}^{\infty} \frac{n\,\Delta p_n}{P_n\,P_{n-1}} + C \sum_{r=1}^{\infty} \frac{P_r}{r\,\chi(r)} \sum_{n=r}^{\infty} \frac{p_n}{P_n\,P_{n-1}} \\ \displaystyle\leqslant C + C \sum_{r=1}^{\infty} \frac{1}{r\,\chi(r)} \leqslant C, \end{cases}$$

by (2.2.2) of Lemma 3 and the hypothesis (iv) of the Theorem.

Finally

$$(2.3.5)\quad \begin{cases} \displaystyle\sum_{n=2}^{\infty} |\gamma_{n,1}| \leqslant \sum_{n=1}^{\infty} P_{n-1} \left[\, \left| \int_0^{\pi} \varphi(t)\,\alpha(t) \cos nt\,dt \right| + \left| \int_0^{\pi} \varphi(t)\,\beta(t) \sin nt\,dt \right| \,\right] \\ \displaystyle\qquad = \sum_{n=1}^{\infty} \frac{|\alpha_n| + |\beta_n|}{P_{n-1}}. \end{cases}$$

Hence to prove the Theorem now it remains to be established that

$$(2.3.6)\qquad \sum_{n=1}^{\infty} \frac{|\alpha_n| + |\beta_n|}{P_{n-1}} < C.$$

It is easy to see that under the hypothesis of the Theorem $\varphi(t)\,\alpha(t) \in L^2$. Thus α_n/π is the Fourier coefficient of an even function which belongs to L^2. The Fourier series of

$$\varphi(t+h)\,\alpha(t+h) - \varphi(t-h)\,\alpha(t-h) \quad \text{being} \quad -(4/\pi) \sum_{n=1}^{\infty} \alpha_n \sin nt \sin nh\,;$$

an application of Parseval's relation gives

$$
(2.3.7)\quad \begin{cases} \sum_{n=1}^{\infty} \alpha_n^2 \sin^2 nh \leqslant C \int_0^{\pi} \{\varphi(t+h)\,\alpha(t+h) - \varphi(t-h)\,\alpha(t-h)\}^2\, dt \\ \leqslant C \int_0^{\pi} \alpha^2(t+h)\,|\varphi(t+h)-\varphi(t-h)|^2\, dt + C \int_{-h}^{h} \varphi^2(t)\,\alpha^2(t+2h)\, dt + \\ \qquad + C \int_{-h}^{h} \varphi^2(t)\,\alpha^2(t)\, dt + C \int_{h}^{\pi} \varphi^2(t)\,|\alpha(t+2h)-\alpha(t)|^2\, dt \\ = \sum_{i=1}^{4} \vartheta_i(h), \text{ say.} \end{cases}
$$

Since $f(x)$ is of bounded variation and $|\varphi(t)|\,\chi(t^{-1}) = O(1)$, it is clear that the function φ is continuous and of bounded variation, and therefore for all positive integral values of N,

$$
(2.3.8)\quad \begin{cases} 2N \int_0^{\pi} \alpha^2\left(t+\frac{\pi}{2N}\right)\left[\varphi\left(t+\frac{\pi}{2N}\right) - \varphi\left(t-\frac{\pi}{2N}\right)\right]^2 dt \\ = \sum_{\nu=1}^{2N} \int_0^{\pi} \alpha^2\left(t+\frac{\nu\pi}{N}\right)\left[\varphi\left(t+\frac{\nu\pi}{N}\right) - \varphi\left(t+\frac{(\nu-1)\pi}{N}\right)\right]^2 dt \\ \leqslant \frac{C}{\chi(N)} \int_0^{\pi} P^2(t^{-1}) \left[\sum_{\nu=1}^{2N} \left|\varphi\left(t+\frac{\nu\pi}{N}\right) - \varphi\left(t+\frac{(\nu-1)\pi}{N}\right)\right|\right] dt \\ \leqslant \frac{C}{\chi(N)}\left[1 + \sum_{k=1}^{\infty} P_k^2\, k^{-2}\right] \leqslant \frac{C}{\chi(N)}, \end{cases}
$$

making use of Lemma 1 and hypothesis (i) of the Theorem.

From (2.3.7) and (2.3.8) it is now clear that

$$
(2.3.9)\qquad \vartheta_1\left(\frac{\pi}{2N}\right) \leqslant \frac{C}{N\,\chi(N)}.
$$

Again, by Lemma 1 and the hypothesis (vi) of the Theorem,

$$
(2.3.10)\qquad \vartheta_2(h) \leqslant C \int_{-h}^{h} \frac{P^2\big((t+2h)^{-1}\big)}{\chi^2(t^{-1})}\, dt \leqslant C\, \frac{h\,P^2(h^{-1})}{\chi^2(h^{-1})},
$$

and

$$
(2.3.11)\qquad \vartheta_3(h) \leqslant C \int_{-h}^{h} \frac{P^2(t^{-1})}{\chi^2(t^{-1})}\, dt \leqslant C\, \frac{h\,P^2(h^{-1})}{\chi^2(h^{-1})}
$$

by the hypothesis (ii) of the Theorem. Also, by Lemma 2,

$$\vartheta_4(h) \leqslant C\, h^2\, P^2(h^{-1}) \int_h^{\pi} \frac{dt}{t^2\, \chi^2(t^{-1})} \leqslant C\, \frac{h\, P^2(h^{-1})}{\chi^2(h^{-1})} \tag{2.3.12}$$

since $t^{\varepsilon}\, \chi(t^{-1})$ increases as t increases.

Combining (2.3.7), (2.3.9), ..., (2.3.12) and taking $N = 2^{\nu}$ and $h = \pi/2^{\nu+1}$, we get

$$\sum_{n=1}^{\infty} \alpha_n^2 \sin^2(n\pi/2^{\nu+1}) \leqslant C \left[\frac{P^2(2^{\nu})}{2^{\nu}\, \chi^2(2^{\nu})} + \frac{1}{2^{\nu}\, \chi(2^{\nu})}\right]. \tag{2.3.13}$$

Applying SCHWARZ's inequality and making use of the above estimate, we have

$$\sum_{n=2}^{\infty} |\alpha_n|/P_n = \sum_{\nu=1}^{\infty} \sum_{n=2^{\nu-1}+1}^{2^{\nu}} |\alpha_n|/P_n \leqslant$$

$$\leqslant C \sum_{\nu=1}^{\infty} [\{ \sum_{n=2^{\nu-1}+1}^{2^{\nu}} P_n^{-2}\}^{1/2} \{ \sum_{n=2^{\nu-1}+1}^{2^{\nu}} \alpha_n^2 \sin^2(n\pi/2^{\nu+1})\}^{1/2}]$$

$$\leqslant C \sum_{\nu=1}^{\infty} (2^{\nu/2}/P(2^{\nu})) \{ \sum_{n=1}^{\infty} \alpha_n^2 \sin^2(n\pi/2^{\nu+1}) \}^{1/2}$$

$$\leqslant C \sum_{\nu=1}^{\infty} \{1/\chi(2^{\nu})\} + C \sum_{\nu=1}^{\infty} \{1/[\chi^{1/2}(2^{\nu})\, P(2^{\nu})]\}$$

$$\leqslant C \sum \{1/[n\, \chi(n)]\} + C \sum \{1/[n\, \chi^{1/2}(n)\, P_n]\} \leqslant C\,,$$

by the application of the conditions (iv) and (v) of the Theorem. Similarly we can show that $\sum |\beta_n|/P_n < \infty$. Thus (2.3.6) is established and the proof of the Theorem is complete.

Remarks. If we choose $p_0 = 1$, $p_n = 0$ $(n \neq 0)$ and $\chi(y) = (\log y)^{2+\varepsilon}$ our theorem yields the well known ZYGMUND's criterion for absolute convergence [9]. Also, choosing $p_n = 1/(n+1)$, we get the following

Theorem A. *If $f(x)$ is of bounded variation and $\chi(y)$ is any one of the functions*

$$(\log y)^{1+\varepsilon},$$

$$\log y \cdot (\log \log y)^{1+\varepsilon},$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\log y \cdot \ldots \cdot \log \log \ldots \log_{p-1} y \cdot (\log \log \ldots \log_p y)^{1+\varepsilon}$$

such that

$$| f(x+t) - f(x) | \, \chi(t^{-1}) < C,$$

then the series $\sum A_n(x)$ is absolutely harmonic summable.

It is interesting to note that a particular case of the above Theorem is known [**8**].

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