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Integrals Involving MacRobert's E-Functions and Integral Function of Two Complex Variables. (**)

1. - Introduction.

Let

$$F(z_1, z_2) = \sum_{n_1, n_2 = 0}^{\infty} \frac{a_{n_1, n_2}}{n_1!\ n_2!} z_1^{n_1} z_2^{n_2}$$

be an integral function of two complex variables z_1 and z_2. Denote

$$M_F(r) = \max_{|z_1| + |z_2| = r} |F(z_1, z_2)|$$

the maximum modulus of $F(z_1, z_2)$.

DŽRBAŠJAN ([1], p. 257) has given the following definition of order:

The integral function $F(z_1, z_2)$ is said to be of finite order ϱ, if

$$\limsup_{r \to \infty} \frac{\log \log M_F(r)}{\log r} = \varrho \qquad (0 \leqslant \varrho < \infty) . \tag{1.1}$$

The obejct of this paper is to evaluate some new type of integrals involving integral functions of two complex variables of order ϱ, based on the properties of MACROBERT'S E-function [1] and the Eulerian integral of the first kind. The particular case when the integral function is of order one has also been studied.

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(1) For definition and properties of E-functions see MACROBERT ([2], p. 352).

2. - We shall use the following two integrals of RATHIE ([3], p. 186):

$$(2.1)\quad \begin{cases} \int_0^\infty t^{2\gamma-1} W_{k,m}(t)\, W_{-k,m}(t)\, E(p;\, \alpha_h : q;\, \beta_s : zt^{-2n})\, dt \\ \qquad = (2\pi)^{\frac{1}{2}-n} (2n)^{2\gamma-\frac{1}{2}} E\{p+4n;\, \alpha_h : q+2n;\, \beta_s : z\,(2n)^{-2n}\}\,, \end{cases}$$

where n is a positive integer, $|\arg z| < \pi$, $\operatorname{Re}(\gamma \pm m + \frac{1}{2}) > 0$,

$$\alpha_{p+\nu} = (2\gamma+\nu)/(2n) \qquad (\nu = 1, 2, \dots, 2n);$$

$$\alpha_{p+2n+i} = (\gamma + m - \tfrac{1}{2} + i)/n\,, \qquad \alpha_{p+3n+i} = (\gamma - m - \tfrac{1}{2} + i)/n\,,$$

$$\beta_{q+i} = (\gamma + k + i)/n\,, \qquad \beta_{q+n+i} = (\gamma - k + i)/n\,,$$

$$(i = 1, 2, \dots, n)$$

and

$$(2.2)\quad \begin{cases} \int_0^\infty t^{2\lambda-1} K_{2\mu}(t)\, K_{2\nu}(t)\, E(p;\, \alpha_h : q;\, \beta_s : zt^{-2n})\, dt \\ \qquad = n^{2\lambda-(3/2)}\, \pi^{(3/2)-n}\, 2^{-n-1} E\{p+4n;\, \alpha_h : q+2n;\, \beta_s : z\, n^{-2n}\}\,, \end{cases}$$

where n is a positive integer, $|\arg z| < \pi$, $\operatorname{Re}(\lambda \pm \mu \pm \nu) > 0$,

$$\alpha_{p+i+1} = (\lambda + \mu + \nu + i)/n\,, \qquad \alpha_{p+n+i+1} = (\lambda - \mu + \nu + i)/n\,,$$

$$\alpha_{p+2n+i+1} = (\lambda + \mu - \nu + i)/n\,, \qquad \alpha_{p+3n+i+1} = (\lambda - \mu - \nu + i)/n\,,$$

$$(i = 0, 1, 2, \dots, n-1);$$

$$\beta_{q+j+1} = (2\lambda + j)/(2n) \qquad (j = 0, 1, 2, \dots, 2n-1)\,.$$

3. - Theorem 1.

Let $|\xi_j| \neq 0$, $|\arg \xi_j| < \dfrac{\pi}{2\varrho}$ $(j = 1, 2)$; *and let*

$$F(z_1, z_2) = \sum_{n_1, n_2 = 0}^{\infty} \frac{a_{n_1,\, n_2}}{n_1!\, n_2!}\, z_1^{n_1} z_2^{n_2}$$

be an integral function of two complex variables z_1 *and* z_2 *of order* ϱ $(0 < \varrho < \infty)$, *then, for* $\arg \xi_1 = \arg \xi_2$, *we have*

$$(3.1)\quad \begin{cases} J_{n_1,n_2}(\xi_1, \xi_2) = \int\limits_0^\infty \int\limits_0^\infty (t_1 \xi_1 + t_2 \xi_2)^{2\gamma\varrho-2} W_{k,m}[(t_1 \xi_1 + t_2 \xi_2)^\varrho] W_{-k,m}[(t_1 \xi_1 + t_2 \xi_2)^\varrho] \cdot \\ \qquad \cdot E[p; \alpha_h : q; \beta_s : z (t_1 \xi_1 + t_2 \xi_2)^{-2n\varrho}] F(t_1, t_2) \, dt_1 \, dt_2 , \end{cases}$$

or

$$(3.2)\quad \begin{cases} J_{n_1,n_2}(\xi_1, \xi_2) = (2\pi)^{\frac{1}{2}-n} (2n)^{2\gamma-\frac{1}{2}} \varrho^{-1} \cdot \\ \qquad \cdot \sum\limits_{n_1,n_2=0}^{\infty} \dfrac{a_{n_1,n_2}}{(n_1+n_2+1)!} \dfrac{(2n)^{(n_1+n_2)\varrho^{-1}} E[p+4n; \alpha_h : q+2n; \beta_s : z (2n)^{-2n}]}{\xi_1^{n_1+1} \xi_2^{n+1}} , \end{cases}$$

where n is a positive integer, $|\arg z| < \pi$, $\operatorname{Re}(\gamma \pm m + \frac{1}{2}) > 0$,

$$\alpha_{p+\nu} = \left(2\gamma + \frac{n_1+n_2}{\varrho} + \nu\right)/(2n) \qquad (\nu = 1, 2, \ldots, 2n) ;$$

$$\alpha_{p+2n+i} = \left(\gamma + \frac{n_1+n_2}{2\varrho} + m - \tfrac{1}{2} + i\right)/n, \quad \alpha_{p+3n+i} = \left(\gamma + \frac{n_1+n_2}{2\varrho} - m - \tfrac{1}{2} + i\right)/n ,$$

$$\beta_{q+i} = \left(\gamma + \frac{n_1+n_2}{2\varrho} + k + i\right)/n , \qquad \beta_{q+n+i} = \left(\gamma + \frac{n_1+n_2}{2\varrho} - k + i\right)/n ,$$

$$(i = 1, 2, \ldots, n) ,$$

and the series in (3.2) *is uniformly and absolutely convergent in a suitably chosen domain.*

Proof. Let us first take $\operatorname{Re} a > 0$, $|\arg z| < \pi$ and consider the integral

$$(3.3)\quad \begin{cases} I_{n_1,n_2}(a) = \int\limits_0^\infty \int\limits_0^\infty (x_1 + x_2)^{2\gamma\varrho-2} W_{k,m}[a (x_1 + x_2)^\varrho] W_{-k,m}[a (x_1 + x_2)^\varrho] \cdot \\ \qquad \cdot E[p; \alpha_h : q; \beta_s : z a^{-2n} (x_1 + x_2)^{-2n\varrho}] x_1^{n_1} x_2^{n_2} dx_1 \, dx_2 , \end{cases}$$

where n, n_1 and n_2 are positive integers. Changing the variables

$$x_1 = t (1-u) , \quad x_2 = t u \qquad (0 \leqslant u \leqslant 1, \quad 0 \leqslant t < +\infty) ,$$

we have

$$I_{n_1,n_2}(a) = \int_0^\infty \int_0^1 t^{n_1+n_2+2\gamma\varrho-2}\, W_{k,m}(at^\varrho)\, W_{-k,m}(at^\varrho) \cdot$$

$$\cdot E[p;\, \alpha_h : q;\, \beta_s : z\, (at^\varrho)^{-2n}] \cdot (1-u)^{n_1}\, u^{n_2}\, \frac{\partial(x_1, x_2)}{\partial(t, u)}\, dt\, du$$

$$= \int_0^\infty \int_0^1 t^{n_1+n_2+2\gamma\varrho-1}\, W_{k,m}(at^\varrho)\, W_{-k,m}(at^\varrho) \cdot$$

$$\cdot E[p;\, \alpha_h : q;\, \beta_s : z\, (at^\varrho)^{-2n}] \cdot (1-u)^{n_1}\, u^{n_2}\, dt\, du\,.$$

Evaluating u-integral with the help of the Eulerian integral of the first kind ([4], p. 212), we obtain

$$I_{n_1,n_2}(a) =$$

$$= \frac{n_1!\, n_2!}{(n_1+n_2+1)!} \int_0^\infty t^{n_1+n_2+2\gamma\varrho-1} W_{k,m}(at^\varrho)\ W_{-k,m}(at^\varrho)\, E[p;\, \alpha_h : q;\, \beta_s : z\, (at^\varrho)^{-2n}]\, dt$$

$$= \frac{n_1!\, n_2!}{\varrho(n_1+n_2+1)!}\, a^{-2\gamma-(n_1+n_2)\varrho^{-1}} \int_0^\infty x^{2\gamma+(n_1+n_2)\varrho^{-1}-1} \cdot$$

$$\cdot W_{k,m}(x)\, W_{-k,m}(x)\, E[p;\, \alpha_h : q;\, \beta_s : z\, x^{-2n}]\, dx\,.$$

Now evaluating x-integral with the help of (2.1), we have

$$(3.4)\quad \left\{ \begin{aligned} I_{n_1,n_2}(a) &= \frac{n_1!\, n_2!}{\varrho(n_1+n_2+1)!} \left\{(2\pi)^{\frac{1}{2}-n}\, (2n)^{2\gamma+(n_1+n_2)\varrho^{-1}-\frac{1}{2}} / a^{2\gamma+(n_1+n_2)\varrho^{-1}}\right\} \cdot \\ &\quad \cdot E[p+4n;\, \alpha_h : q+2n;\, \beta_s : z\, (2n)^{-2n}]\,, \end{aligned} \right.$$

where:

$$\mathrm{Re}\,(\gamma \pm m + \tfrac{1}{2}) > 0\,, \qquad \alpha_{p+\nu} = \left(2\gamma + \frac{n_1+n_2}{\varrho} + \nu\right)/(2n) \qquad (\nu = 1, 2, \ldots, 2n)\,;$$

$$\alpha_{p+2n+i} = \left(\gamma + \frac{n_1+n_2}{2\varrho} + m - \tfrac{1}{2} + i\right)/n\,, \qquad \alpha_{p+3n+i} = \left(\gamma + \frac{n_1+n_2}{2\varrho} - m - \tfrac{1}{2} + i\right)/n\,,$$

$$\beta_{q+i} = \left(\gamma + \frac{n_1+n_2}{2\varrho} + k + i\right)/n\,, \qquad \beta_{q+n+i} = \left(\gamma + \frac{n_1+n_2}{2\varrho} - k + i\right)/n\,,$$

$$(i = 1, 2, \ldots, n)\,.$$

Let $\arg\xi_1 = \arg\xi_2 = \alpha$ and if we denote $a = e^{i\alpha\varrho}$ where $\operatorname{Re} a = \cos(\alpha\varrho) > 0$, $|\arg z| < \pi$, then from (3.1) we obtain

$$
(3.5)\quad \begin{cases}
J_{n_1,n_2}(\xi_1, \xi_2) = \int_0^\infty\int_0^\infty (t_1\xi_1 + t_2\xi_2)^{2\gamma\varrho-2} W_{k,m}[(t_1\xi_1 + t_2\xi_2)^\varrho]\, W_{-k,m}[(t_1\xi_1 + t_2\xi_2)^\varrho]\cdot \\
\qquad \cdot E[p;\alpha_h:q;\beta_s:z(t_1\xi_1 + t_2\xi_2)^{-2n\varrho}]\left(\sum_{n_1,n_2=0}^{\infty} \frac{a_{n_1,n_2}}{n_1!\,n_2!} t_1^{n_1} t_2^{n_2}\right) dt_1\, dt_2 \\
= e^{2i\alpha(\gamma\varrho-1)} \sum_{n_1,n_2=0}^{\infty} \frac{a_{n_1,n_2}}{n_1!\,n_2!} \int_0^\infty\int_0^\infty (t_1|\xi_1| + t_2|\xi_2|)^{2\gamma\varrho-2} W_{k,m}[a(t_1|\xi_1| + t_2|\xi_2|)^\varrho]\cdot \\
\qquad \cdot W_{-k,m}[a(t_1|\xi_1| + t_2|\xi_2|)^\varrho] E[p;\alpha_h:q;\beta_s:z a^{-2n}(t_1|\xi_1| + t_2|\xi_2|)^{-2n\varrho}] t_1^{n_1} t_2^{n_2} dt_1 dt_2 .
\end{cases}
$$

Replacing $t_1|\xi_1|$ by x_1 and $t_2|\xi_2|$ by x_2 and evaluating the integrals, we get

$$
J_{n_1,n_2}(\xi_1, \xi_2) = e^{2i\alpha(\gamma\varrho-1)} \sum_{n_1,n_2=0}^{\infty} \frac{a_{n_1,n_2}}{n_1!\,n_2!} \frac{1}{|\xi_1|^{n_1+1}\,|\xi_2|^{n_2+1}} \int_0^\infty\int_0^\infty (x_1 + x_2)^{2\gamma\varrho-2} W_{k,m}[a(x_1 + x_2)^\varrho]\cdot
$$

$$
\cdot W_{-k,m}[a(x_1 + x_2)^\varrho]\, E[p;\alpha_h:q;\beta_s:z\,a^{-2n}(x_1 + x_2)^{-2n\varrho}]\, x_1^{n_1} x_2^{n_2} dx_1\, dx_2 =
$$

$$
= (2\pi)^{\frac{1}{2}-n}(2n)^{2\gamma-\frac{1}{2}} e^{2i\alpha(\gamma\varrho-1)} \varrho^{-1}\cdot \sum_{n_1,n_2=0}^{\infty} \frac{a_{n_1,n_2}}{(n_1 + n_2 + 1)!} \{(2n)^{(n_1+n_2)\varrho^{-1}}/a^{2\gamma+(n_1+n_2)\varrho^{-1}}\}\cdot
$$

$$
\cdot\{E[p + 4n;\alpha_h:q + 2n;\beta_s:z(2n)^{-2n}]/(|\xi_1|^{n_1+1}|\xi_2|^{n_2+1})\},
$$

due to the relation (3.4).

Thus under the conditions $|\xi_j| \neq 0$, $\arg\xi_1 = \arg\xi_2$, $|\arg\xi_j| < \pi/(2\varrho)$, $(j = 1, 2)$, and an appeal to analytic continuation, we obtain

$$
J_{n_1,n_2}(\xi_1, \xi_2) = \{(2\pi)^{\frac{1}{2}-n}(2n)^{2\gamma-\frac{1}{2}}\}\,\varrho^{-1} \sum_{n_1,n_2=0}^{\infty} \frac{a_{n_1,n_2}}{(n_1 + n_2 + 1)!} (2n)^{(n_1+n_2)\varrho^{-1}}\cdot
$$

$$
\cdot E[p + 4n;\alpha_h:q + 2n;\beta_s:z(2n)^{-2n}]/(\xi^{n_1+1}\xi^{n_2+1}),
$$

provided the change of order of integration and summation in (3.5) is justified.

Regarding the change of order of integration and summation in (3.5), we note that $F(z_1, z_2)$ is an integral function of the variables z_1 and z_2 and so the series

$$
\sum_{n_1,n_2=0}^{\infty} \frac{a_{n_1,n_2}}{n_1!\,n_2!} z_1^{n_1} z_2^{n_2}
$$

converges uniformly for $|z_j| < R_j$ $(j = 1, 2)$ for R_j arbitrary large, the integrals converge uniformly and absolutely under the conditions imposed in the theorem and since the resulting series

$$\sum_{n_1,n_2=0}^{\infty} \frac{a_{n_1,n_2}}{(n_1+n_2+1)!} (2n)^{(n_1+n_2)\varrho-1} E[p+4n;\, \alpha_h : q+2n;\, \beta_s : z\,(2n)^{-2n}]/(\xi_1^{n_1+1}\xi_2^{n_2+1})$$

converges uniformly and absolutely, therefore, the change of order of integration and summation in (3.5) is justified.

Particular Case. If we take $\varrho = 1$, then the above theorem reduces to the following result:

Let $|\xi_j| \neq 0$, $|\arg \xi_j| < \pi/2$, $(j = 1,2)$; and let

$$F(z_1, z_2) = \sum_{n_1,n_2=0}^{\infty} \frac{a_{n_1,n_2}}{n_1!\, n_2!} z_1^{n_1} z_2^{n_2}$$

be an integral function of two complex variables z_1 and z_2, then for $\arg \xi_1 = \arg \xi_2$ we have

$$(3.6)\quad \begin{cases} J_{n_1,n_2}(\xi_1, \xi_2) = \int_0^\infty \int_0^\infty (t_1\xi_1 + t_2\xi_2)^{2\gamma-2}\, W_{k,m}[t_1\xi_1 + t_2\xi_2]\, W_{-k,m}[t_1\xi_1 + t_2\xi_2] \cdot \\ \qquad \cdot E[p;\, \alpha_h : q;\, \beta_s : z\,(t_1\xi_1 + t_2\xi_2)^{-2n}]\, F(t_1, t_2)\, dt_1\, dt_2 = \\ = (2\pi)^{\frac{1}{2}-n} (2n)^{2\gamma-\frac{1}{2}} \sum_{n_1,n_2=0}^{\infty} \frac{a_{n_1,n_2}}{(n_1+n_2+1)!} (2n)^{n_1+n_2} \cdot \\ \qquad \cdot E[p+4n;\, \alpha_h : q+2n;\, \beta_s : z\,(2n)^{-2n}]/(\xi_1^{n_1+1}\xi_2^{n_2+1}), \end{cases}$$

where n is a positive integer, $|\arg z| < \pi$, $\operatorname{Re}(\gamma \pm m + \frac{1}{2}) > 0$,

$$\alpha_{p+\nu} = (2\gamma + n_1 + n_2 + \nu)/2n) \qquad (\nu = 1, 2, \ldots, 2n);$$

$$\alpha_{p+2n+i} = \left(\gamma + \frac{n_1+n_2}{2} + m - \tfrac{1}{2} + i\right)/n, \quad \alpha_{p+3n+i} = \left(\gamma + \frac{n_1+n_2}{2} - m - \tfrac{1}{2} + i\right)/n,$$

$$\beta_{q+i} = \left(\gamma + \frac{n_1+n_2}{2} + k + i\right)/n, \qquad \beta_{q+n+i} = \left(\gamma + \frac{n_1+n_2}{2} - k + i\right)/n,$$

$$(i = 1, 2, \ldots, n),$$

and the series in (3.6) converges uniformly and absolutely.

4. - Theorem 2.

Let $|\xi_l| \neq 0$, $|\arg \xi_l| < \dfrac{\pi}{2\varrho}$, $(l = 1, 2)$; *and let*

$$F(z_1, z_2) = \sum_{n_1, n_2 = 0}^{\infty} \frac{a_{n_1, n_2}}{n_1!\; n_2!} z_1^{n_1} z_2^{n_2}$$

be an integral function of two complex variables z_1 *and* z_2 *of order* ϱ $(0 < \varrho < \infty)$, *then for* $\arg \xi_1 = \arg \xi_2$ *we have*

$$(4.1)\quad \begin{cases} P_{n_1,n_2}(\xi_1, \xi_2) = \int_0^\infty \int_0^\infty (t_1 \xi_1 + t_2 \xi_2)^{2\lambda\varrho - 2} K_{2\mu}[(t_1 \xi_1 + t_2 \xi_2)^\varrho] K_{2\nu}[(t_1 \xi_1 + t_2 \xi_2)^\varrho] \cdot \\ \qquad \cdot E[p;\, \alpha_h : q;\, \beta_s : z\, (t_1 \xi_1 + t_2 \xi_2)^{-2n\varrho}] F(t_1, t_2)\, \mathrm{d}t_1 \mathrm{d}t_2 , \end{cases}$$

or

$$(4.2)\quad \begin{cases} P_{n_1,n_2}(\xi_1, \xi_2) = 2^{-n-1} \pi^{(3/2)-n} n^{2\lambda-(3/2)} \varrho^{-1} \sum_{n_1,n_2=0}^{\infty} \dfrac{a_{n_1,n_2}}{(n_1 + n_2 + 1)!} \cdot \\ \qquad \cdot n^{(n_1+n_2)\varrho^{-1}} E[p + 4n;\, \alpha_h : q + 2n;\, \beta_s : zn^{-2n}] / (\xi_1^{n_1+1} \xi_2^{n_2+1}), \end{cases}$$

where n *is a positive integer,* $|\arg z| < \pi$, $\mathrm{Re}(\lambda \pm \mu \pm \nu) > 0$,

$$\alpha_{p+i+1} = \left(\lambda + \frac{n_1 + n_2}{2\varrho} + \mu + \nu + i\right)/n, \quad \alpha_{p+n+i+1} = \left(\lambda + \frac{n_1 + n_2}{2\varrho} - \mu + \nu + i\right)/n,$$

$$\alpha_{p+2n+i+1} = \left(\lambda + \frac{n_1 + n_2}{2\varrho} + \mu - \nu + i\right)/n, \quad \alpha_{p+3n+i+1} = \left(\lambda + \frac{n_1 + n_2}{2\varrho} - \mu - \nu + i\right)/n,$$

$$(i = 0, 1, 2, \ldots, n-1),$$

$$\beta_{q+j+i} = \left(2\lambda + \frac{n_1 + n_2}{\varrho} + j\right)/(2n) \qquad (j = 0, 1, 2, \ldots, 2n-1),$$

and the series in (4.2) *is uniformly and absolutely convergent in a suitably chosen domain.*

Proof. The proof is similar to that of Theorem 1, except that we use integral (2.2) instead of (2.1).

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References.

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