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Two Inversion Integrals. (**)

1. – TA LI [4] gave inversion integral for an integral transformation which involves a CHEBYSHEV polynomial in the Kernel. A similar problem involving LEGENDRE polynomial in the Kernel is solved by BUSCHMAN [1]. D.V. WIDDER [5] applied the methods of LAPLACE-transformation to solve such problems. In [2] we have solved the integral equation with BESSEL functions in the Kernel with a different method. Applying the same method in the present paper we solve two integral equations with modified BESSEL function $K_\nu(x)$ and the STRUVE's function $H_\nu(x)$ in the Kernel.

Modified BESSEL function $K_\nu(x)$ is defined as

$$K_\nu(x) = \frac{\pi}{2}\,\frac{I_{-\nu}(x) - I_\nu(x)}{\sin(\nu\pi)} \tag{1.1}$$

and the STRUVE's function $H_\nu(x)$ as

$$H_\nu(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{\nu+2m+1}}{\Gamma\big(m+(3/2)\big)\ \Gamma\big(\nu+m+(3/2)\big)}\,. \tag{1.2}$$

2. - Results required in the proof.

If $K_\nu(x)$ and $I_\nu(x)$ are the modified BESSEL functions, then the K-transform

$$\int_0^\infty f(x)\ K_\nu(xy)\ (xy)^{1/2}\ \mathrm{d}x = g(y) \tag{2.1}$$

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has the inversion formula

$$(2.2)\qquad \frac{1}{\pi i}\int_{c-i\infty}^{c+i\infty} g(y)\, I_\nu(xy)\,(xy)^{1/2}\,\mathrm{d}y = f(x)\,.$$

Similarly the H-transform, defined as

$$(2.3)\qquad \int_0^\infty f(x)\, H_\nu(xy)\,(xy)^{1/2}\,\mathrm{d}x = g(y)$$

has the inversion formula

$$(2.4)\qquad \int_0^\infty g(y)\, Y_\nu(xy)\,(xy)^{1/2}\,\mathrm{d}x = f(x)\,, \qquad -1/2 < \nu < 1/2\,.$$

Here $Y_\nu(x)$ is the WEBER'S BESSEL function of second kind and order ν.

Now in [3] (p. 209, (59)) putting $\mu = 1$ and replacing a by $z\sqrt{y}$ we get

$$(2.5)\qquad \int_y^\infty x^{-\nu/2}\, K_\nu(z\sqrt{xy})\,\mathrm{d}x = \frac{2}{z}\, y^{-\nu/2}\, K_{\nu-1}(zy)\,.$$

Similarly from [3] (p. 199, (88)) we get

$$(2.6)\qquad \int_0^y x^{\nu/2}\, K_\nu(z\sqrt{xy}\,)\,\mathrm{d}x = \frac{2}{z}\, y^{\nu/2}\, H_{\nu+1}(zy)\,, \qquad \nu > -3/2\,.$$

3. - Theorem I.

If the integral equations

$$(3.1)\qquad \int_0^x K_\nu(z\sqrt{xy})\, f(y)\,\mathrm{d}y = \varphi(x,\, z)$$

and

$$(3.2)\qquad \int_0^\infty x^{-\nu/2}\,\varphi(x,\, z)\,\mathrm{d}x = \psi(z)$$

exist, then the solution of (3.1) *is given by*

$$(3.3)\qquad f(y) = \frac{y^{1+(\nu/2)}}{2\pi i}\int_{c-i\infty}^{c+i\infty} I_{\nu-1}(zy)\,\{z^2\,\psi(z)\}\,\mathrm{d}z\,.$$

The inversion formula (3.3) is in the terms of function $\psi(z)$ and to find $\psi(z)$ is not difficult since it is an easy transformation of the known function $\varphi(x, z)$.

Suppose $\varphi(x, z)$ and $\psi(z)$ both exist, then putting the value of $\varphi(x, z)$ in (3.2) from (3.1) we have

$$\int_0^\infty x^{-\nu/2}\Big\{\int_0^x K_\nu(z\sqrt{xy})\, f(y)\, dy\Big\}\, dx = \psi(z)\,, \tag{3.4}$$

changing the order of integration we get

$$\int_0^\infty f(y)\Big\{\int_y^\infty x^{-\nu/2}\, K_\nu(z\sqrt{xy})\, dx\Big\}\, dy = \psi(z)\,, \tag{3.5}$$

making use of the result (2.5) we get

$$2\int_0^\infty y^{-\nu/2}\, K_{\nu-1}(zy)\, f(y)\, dy = z\,\psi(z)\,.$$

Now writing this in the following form

$$\int_0^\infty \{y^{-(\nu+\frac{1}{2})} f(y)\}\sqrt{zy}\, K_{\nu-1}(zy)\, dy = \frac{1}{2}\, z^{3/2}\, \psi(z)$$

and making use of (2.1) we get (3.3).

4. - Theorem II.

If the integral equations

$$\int_x^\infty H_\nu(z\sqrt{xy})\, g(y)\, dy = \varphi_1(x, z)\,, \qquad -1/2 > \nu > -3/2\,, \tag{4.1}$$

and

$$\int_0^\infty x^{\nu/2}\, \varphi_1(x, z)\, dx = \psi_1(z) \tag{4.2}$$

exist, then the solution of (4.1) *is given by*

$$g(y) = \frac{1}{2}\, y^{1-(\nu/2)} \int_0^\infty Y_{\nu+1}(yz)\, \{z^2\, \psi_1(z)\}\, dz\,. \tag{4.3}$$

As in previous problem putting the value of $\varphi_1(x, z)$ in (4.2) from (4.1), we have

$$\int_0^\infty x^{\nu/2} \left\{ \int_x^\infty H_\nu(z\sqrt{xy})\, g(y)\, dy \right\} dx = \psi_1(z) \,,$$

changing the order of integration we get

$$\int_0^\infty \left\{ \int_0^y x^{\nu/2} H_\nu(z\sqrt{xy})\, dx \right\} g(y)\, dy = \psi_1(z) \,.$$

Again making use of (2.6) we have

$$\int_0^\infty y^{\nu/2} H_{\nu+1}(zy)\, g(y)\, dy = \frac{1}{2} z\, \psi_1(z) \,.$$

Writing this in the form of (2.3) and applying the inversion formula (2.4) we get (4.3).

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