Riv. Mat. Univ. Parma (2) 10 (1969), 171-179

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On the Maximum Term and the Proximate Order of an Entire Function. (**)

1. – Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be an entire function of order ϱ $(0 < \varrho < \infty)$. A real-valued, continuous and piecewise differentiable function $\varrho(r)$ is called a proximate order ([1], p. 32) if it satisfies the conditions

$$\lim_{r \to \infty} \varrho(r) = \varrho\ , \tag{1.1}$$

$$\lim_{r \to \infty} \big(r\ \varrho'(r) \log r\big) = 0, \tag{1.2}$$

where $\varrho'(r)$ is either the right or left-hand derivative at points where they are different.

If, for the entire function $f(z)$, the quantity

$$T = \limsup_{r \to \infty} \big(r^{-\varrho(r)} \log M(r)\big), \qquad \text{where} \quad M(r) = \max_{|z|=r} |\,f(z)\,|\ , \tag{1.3}$$

is different from zero and infinity, then $\varrho(r)$ is called a proximate order of the given entire function $f(z)$, and T is called the type of the function $f(z)$ with respect to the proximate order $\varrho(r)$. If the limit exists in (1.3), then we say that $f(z)$ is of perfectly regular growth with respect to the proximate order $\varrho(r)$.

Let $\mu(r)$ be the maximum term of rank $\nu(r)$ in the entire series for $f(z)$ for

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$|z| = r$ so that we have

$$\mu(r) = \max_{n\geq 0}\{\,|a_n|\,r^n\}, \qquad \nu(r) = \max\{n \mid \mu(r) = |a_n|\,r^n\}.$$

It is known ([2], p. 31) that

$$\log\mu(r) = \log\mu(r_0) + \int_{r_0}^{r} x^{-1}\,\nu(x)\,\mathrm{d}x, \qquad 0 < r_0 < r. \tag{1.4}$$

In the present paper we obtain another criteria for determining whether an arbitrary proximate order $\varrho(r)$ is a proximate order of the entire function $f(z)$. We then obtain a necessary and sufficient condition for $f(z)$ to be of perfectly regular growth with respect to its proximate order $\varrho(r)$. We also obtain a number of relations involving $\nu(r)$ and $\varrho(r)$.

2. – We start by proving a lemma.

L e m m a. *If $\varrho(r)$ is a proximate order satisfying* (1.1) *and* (1.2), *then* (1)

$$\int_{r_0}^{r} t^{\varrho(t)-1}\,\mathrm{d}t \sim \frac{r^{\varrho(r)}}{\varrho} \qquad \textit{as}\ \ r\to\infty. \tag{2.1}$$

P r o o f. Integrating by parts, we have

$$\int_{r_0}^{r} t^{\varrho(t)-1}\,\mathrm{d}t = \int_{r_0}^{r} t^{\varrho(t)-\varrho}\cdot t^{\varrho-1}\,\mathrm{d}t$$

$$= \frac{t^{\varrho(t)}}{\varrho}\Bigg|_{r_0}^{r} - \frac{1}{\varrho}\int_{r_0}^{r} t^{\varrho(t)-1}\{\varrho(t)-\varrho+t\,\varrho'(t)\log t\}\,\mathrm{d}t\,.$$

Since $\varrho(r)$ satisfies (1.1) and (1.2), we have

$$\varrho(t) = \varrho + o(1)\,, \qquad t\,\varrho'(t)\log t = o(1)\,.$$

So, we have

$$\big(1+o(1)\big)\int_{r_0}^{r} t^{\varrho(t)-1}\,\mathrm{d}t = \frac{r^{\varrho(r)}}{\varrho} + O(1)\,,$$

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(1) r_0 is a positive constant which need not be the same at each occurrence in the present paper.

which gives

$$\int_{r_0}^{r} t^{\varrho(t)-1}\, dt \sim \frac{r^{\varrho(r)}}{\varrho} \qquad \text{as} \quad r \to \infty .$$

Hence the Lemma.

Theorem 1. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *be an entire function of order* ϱ $(0 < \varrho < \infty)$ *and let* $\varrho(r)$ *be an arbitrary proximate order. Then* $\varrho(r)$ *is a proximate order of the entire function* $f(z)$ *if and only if*

$$0 < \gamma = \limsup_{r=\infty} \left(r^{-\varrho(r)} \nu(r)\right) < \infty , \tag{2.2}$$

where $\nu(r)$ *denotes the rank of the maximum term* $\mu(r)$ *in the entire series for* $f(z)$ *for* $|z| = r$.

Proof. Since $f(z)$ is an entire function of finite order, we have ([2], p.32)

$$\log M(r) \sim \log \mu(r) \qquad \text{as } r \to \infty,$$

so, if $\varrho(r)$ is a proximate order of $f(z)$, (1.3) gives

$$0 < \limsup_{r \to \infty} \left(r^{-\varrho(r)} \log \mu(r)\right) = T < \infty . \tag{2.3}$$

Now, if $k \geqslant 1$ and $\liminf_{r \to \infty} \left(r^{-\varrho(r)} \nu(r)\right) = \delta$, we have

$$\log \mu(kr) = O(1) + \int_{r_0}^{r} t^{-1} \nu(t)\, dt + \int_{r}^{kr} t^{-1} \nu(t)\, dt > \int_{r_0}^{r} (\delta - \varepsilon)\, t^{\varrho(t)-1}\, dt + \nu(r) \log k ,$$

or

$$\log \mu(kr) > \frac{(\delta - \varepsilon) r^{\varrho(r)}}{\varrho} + \nu(r) \log k , \tag{2.4}$$

in view of (2.1). Dividing both sides of (2.4) by $(kr)^{\varrho(kr)}$, we get

$$\frac{\log \mu(kr)}{(kr)^{\varrho(kr)}} > \frac{\delta - \varepsilon}{\varrho} \frac{r^{\varrho(r)}}{(kr)^{\varrho(kr)}} + \frac{\nu(r)}{r^{\varrho(r)}} \frac{r^{\varrho(r)}}{(kr)^{\varrho(kr)}} \log k . \tag{2.5}$$

Proceeding to limits and making use of the result ([1], p. 33)

$$(kr)^{\varrho(kr)} \sim k^{\varrho}\, r^{\varrho(r)} \qquad \text{as } r \to \infty \tag{2.6}$$

for every k satisfying $0 < k < \infty$, we get

$$T \geqslant \frac{\delta + \gamma \varrho \log k}{\varrho\, k^{\varrho}}\,. \tag{2.7}$$

Also, we have, if $\gamma < \infty$,

$$\begin{cases} \log \mu(kr) < O(1) + \int_{r_0}^{r} (\gamma + \varepsilon)\; t^{\varrho(t)-1}\, dt + \nu(kr) \log k \\ \qquad \sim \dfrac{(\gamma + \varepsilon)\, r^{\varrho(r)}}{\varrho} + \nu(kr) \log k, \end{cases} \tag{2.8}$$

in view of (2.1). Dividing throughout by $(kr)^{\varrho(kr)}$, proceeding to limits and making use of (2.6), we get

$$T \leqslant \frac{\gamma\; (1 + \varrho\, k^{\varrho} \log k)}{\varrho\, k^{\varrho}}\,. \tag{2.9}$$

Now, if $T > 0$, (2.9) gives $\gamma > 0$ while if $T < \infty$ by (2.7) we have $\gamma < \infty$ so that if $0 < T < \infty$ we have $0 < \gamma < \infty$. On the other hand, if $\gamma > 0$, (2.7) gives $T > 0$ while by (2.9) $\gamma < \infty$ implies $T < \infty$ so that if $0 < \gamma < \infty$ then $0 < T < \infty$. Hence the theorem.

Theorem 2. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *be an entire function of order* ϱ $(0 < \varrho < \infty)$. *Then* $f(z)$ *is of perfectly regular growth and type* T *with respect to its proximate order* $\varrho(r)$ *if and only if*

$$\nu(r) \sim \varrho\, T\, r^{\varrho(r)} \qquad \textit{as } r \to \infty, \tag{2.10}$$

where $\nu(r)$ *denotes the rank of the maximum term* $\mu(r)$ *in the entire series for* $f(z)$ *for* $|z| = r$.

Proof. Let

$$\lim_{r \to \infty} \{ r^{-\varrho(r)} \log \mu(r) \} = T\,,$$

so that, for $r > r_0 = r_0(\varepsilon)$,

$$r^{\varrho(r)}(T-\varepsilon) < \log\mu(r) < r^{\varrho(r)}(T+\varepsilon). \tag{2.11}$$

Now, if $a > 0$,

$$\int_r^{r(1+a)} x^{-1}\nu(x)\,dx = \int_0^{r(1+a)} x^{-1}\nu(x)\,dx - \int_0^r x^{-1}\nu(x)\,dx =$$

$$= \log\mu(r(1+a)) - \log\mu(r) < (T+\varepsilon)\, r^{\varrho(r)}\,(1+a)^{\varrho(r)} - (T-\varepsilon)\, r^{\varrho(r)},$$

in view of (2.11). But

$$\int_r^{r(1+a)} x^{-1}\nu(x)\,dx \geqslant \nu(r)\int_r^{r(1+a)} x^{-1}\,dx > \frac{\nu(r)\,a}{1+a},$$

so that

$$\frac{\nu(r)\,a}{1+a} < (T+\varepsilon)(1+a)^{\varrho(r)}\, r^{\varrho(r)} - (T-\varepsilon)\, r^{\varrho(r)}$$

or

$$\frac{\nu(r)}{r^{\varrho(r)}} < (T+\varepsilon)\,\frac{1+a}{a}\,(1+a)^{\varrho(r)} - (T-\varepsilon)\,\frac{1+a}{a}$$

$$= T\varrho(r)\left\{1 + a + o(a^2)\right\} + \varepsilon\,\frac{1+a}{a}\left\{(1+a)^{\varrho(r)} + 1\right\}.$$

Since a is arbitrary, we get

$$\limsup_{r\to\infty} \frac{\nu(r)}{r^{\varrho(r)}} \leqslant \varrho T.$$

Similarly, considering $\int_{r(1-a)}^{r} x^{-1}\nu(x)\,dx$ and proceeding as above, we get

$$\liminf_{r\to\infty} \frac{\nu(r)}{r^{\varrho(r)}} \geqslant \varrho T.$$

Hence

$$\lim_{r\to\infty} \frac{\nu(r)}{r^{\varrho(r)}} = \varrho T.$$

Now, let

$$\lim_{r\to\infty} \frac{\nu(r)}{r^{\varrho(r)}} = \varrho T ,$$

then, for $r > r_0' = r_0'(\varepsilon)$,

$$(\varrho T - \varepsilon)\, r^{\varrho(r)} < \nu(r) < (\varrho T + \varepsilon)\, r^{\varrho(r)} . \tag{2.12}$$

Differentiating the relation (1.4) with respect to r, we get for *almost all values* of r,

$$\mu'(r)/\mu(r) = \nu(r)/r ,$$

where $\mu'(r)$ denotes the derivative of $\mu(r)$. Substituting in (2.12), we get, for almost all $r > r_0'$,

$$(\varrho T - \varepsilon)\, r^{\varrho(r)-1} < \mu'(r)/\mu(r) < (\varrho T + \varepsilon)\, r^{\varrho(r)-1} .$$

Integrating between the limits r_0' to r and making use of (2.1), we get, for *all* $r > r_0'$,

$$\left(T - \frac{\varepsilon}{\varrho}\right) r^{\varrho(r)} < \log \mu(r) < \left(T + \frac{\varepsilon}{\varrho}\right) r^{\varrho(r)} ,$$

which gives

$$\lim_{r\to\infty} \frac{\log \mu(r)}{r^{\varrho(r)}} = T.$$

This prove the theorem.

3. – Let $\varrho(r)$ be a proximate order of the entire function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ of order ϱ $(0 < \varrho < \infty)$ and let

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log M(r)}{r^{\varrho(r)}} = \lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log \mu(r)}{r^{\varrho(r)}} = \frac{T}{t} , \tag{3.1}$$

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\nu(r)}{r^{\varrho(r)}} = \frac{\gamma}{\delta} . \tag{3.2}$$

In the present section we derive various relations between the constants defined above. We first prove

Theorem 3. *If* $\varrho(r)$ *be a proximate order of the entire function* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *of order* ϱ $(0 < \varrho < \infty)$ *and* T, t, γ, δ *are defined as in* (3.1) *and* (3.2), *then*

$$\delta \leqslant \frac{\gamma}{e} e^{\delta/\gamma} \leqslant \varrho T \leqslant \gamma \,, \tag{3.3}$$

$$\delta \leqslant \varrho t \leqslant \delta \left(1 + \log \frac{\gamma}{\delta}\right) \leqslant \gamma \,. \tag{3.4}$$

Proof. Proceeding to limits in (2.5), we get, in view of (3.1) and (3.2),

$$T \geqslant \frac{\delta + \gamma\, \varrho \log k}{\varrho\, k^{\varrho}} \,, \tag{3.5}$$

$$t \geqslant \frac{\delta\,(1 + \varrho \log k)}{\varrho\, k^{\varrho}} \,. \tag{3.6}$$

Taking $k = 1$ in (3.6) and $k = \exp\{(\gamma - \delta)/(\gamma\delta)\}$ in (3.5), we get

$$t \geqslant \delta/\varrho \,, \qquad e\varrho T \geqslant \gamma\, e^{\delta/\gamma} \geqslant e\delta \,,$$

since $\exp x \geqslant ex$ for $x \geqslant 0$. Further, dividing (2.8) by $(kr)^{\varrho(kr)}$ proceeding to limits and making use of (2.6), we get

$$T \leqslant \frac{\gamma\,(1 + \varrho\, k^{\varrho} \log k)}{\varrho\, k^{\varrho}} \,, \tag{3.7}$$

$$t \leqslant \frac{\gamma + \varrho\, \delta\, k^{\varrho} \log k}{\varrho\, k^{\varrho}} \,. \tag{3.8}$$

Taking $k = 1$ in (3.7) and $k = (\gamma/\delta)^{1/\varrho}$ in (3.8), we get

$$T \leqslant \gamma/\delta \,, \qquad \varrho\, t \leqslant \delta \left(1 + \log \frac{\gamma}{\delta}\right) \leqslant \delta \frac{\gamma}{\delta} = \gamma \,,$$

since $\log(1 + x) \leqslant x$ for $x \geqslant 0$. This proves the theorem.

Remark. Since $e^x \geqslant 1 + x$ for $x \geqslant 0$, we get, from (3.3), $\gamma + \delta \leqslant e\varrho T$. This inequality can be further improved as is shown in the following theorem.

Theorem 4. *If the constants have the meaning as before, then*

$$\gamma + \varrho\, t \leqslant e\varrho T\,, \tag{3.9}$$

$$e\,\varrho\, t \leqslant eT + e\delta\,. \tag{3.10}$$

Proof. We have, if $k = e^{1/\varrho}$,

$$\log\mu(kr) = \log\mu(r) + \int_r^{kr} x^{-1}\,\nu(x)\,\mathrm{d}x > (t - \varepsilon)\, r^{\varrho(r)} + \frac{\nu(r)}{\varrho} \qquad \text{for } r > r_0\,,$$

$$\frac{\log\mu(kr)}{(kr)^{\varrho(kr)}} > (t - \varepsilon)\,\frac{r^{\varrho(r)}}{(kr)^{\varrho(kr)}} + \frac{1}{\varrho}\,\frac{\nu(r)}{r^{\varrho(r)}}\,\frac{r^{\varrho(r)}}{(kr)^{\varrho(kr)}}\,.$$

Proceeding to limits and using (2.6), we get

$$T \geqslant \frac{t}{e} + \frac{\gamma}{e\varrho}\,,$$

which gives (3.9). Further,

$$\log\mu(kr) \leqslant \log\mu(r) + \frac{\nu(kr)}{\varrho} < (T + \varepsilon)\, r^{\varrho(r)} + \frac{\nu(kr)}{\varrho}\,, \qquad r > r_0\,,$$

$$\frac{\log\mu(kr)}{(kr)^{\varrho(kr)}} < (T + \varepsilon)\,\frac{r^{\varrho(r)}}{(kr)^{\varrho(kr)}} + \frac{1}{\varrho}\,\frac{\nu(kr)}{(kr)^{\varrho(kr)}}\,.$$

Again proceeding to limits and making use of (2.6) once again, we get, finally,

$$t \leqslant \frac{T}{e} + \frac{\delta}{\varrho} \qquad \text{i.e.} \qquad e\varrho\, t \leqslant \varrho T + e\delta\,,$$

which is (3.10).

Remark. Since, by (3.4), $\delta \leqslant \varrho\, t$, (3.9) is a refinement of the inequality $\gamma + \delta \leqslant e\varrho T$.

References.

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[2] G. VALIRON, *Lectures on the General Theory of Integral Functions*, Chelsea Publ. Co., New York 1949.

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