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Note on Torsional Vibrations and Stresses in a Non-Homogeneous Elastic Cylinder. (**)

1. - Introduction.

A tendency to extend results obtained in homogeneous elastic bodies to those in non-homogeneous cases has developed very recently. In the problem under consideration we have shown that the results of G. CINELLI [1] who has found solution in the homogeneous case by the application of finite HANKEL transforms [2] can be extended to the non-homogeneous case also on considering the modulus of rigidity varying as the square of the distance from the axis of the cylinder.

To make the paper more or less self contained we cite here some of the important properties of finite HANKEL transforms from [1]:

(1) $$\bar{f}(\xi_i) = H[f(r)] = \int_a^b r\, f(r)\, C_m(r,\ \xi_i)\, \mathrm{d}r\,, \qquad a \leqslant r \leqslant b\,,$$

(2) $$C_m(r,\ \xi_i) = \{J_m(\xi_i r)\,[\xi_i Y'_m(\xi_i a) + h Y_m(\xi_i a)] - Y_m(\xi_i r)\,[\xi_i J'_m(\xi_i a) + h J_m(\xi_i a)]\}\,,$$

(3) $$f(r) = \frac{\pi^2}{2} \sum \xi_i^2\,[\xi_i J'_m(\xi_i b) + k J_m(\xi_i b)]^2\, \bar{f}(\xi_i)\, \frac{C_m(r,\ \xi_i)}{F_m(\xi_i)}\,,$$

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$$F_m(\xi_i) = \left\{k^2 + \xi_i^2\left[1-\left(\frac{m}{\xi_i b}\right)^2\right]\right\}[\xi_i J'_m(\xi_i a) + h J_m(\xi_i a)]^2 - \left\{h^2 + \xi_i^2\left[1-\left(\frac{m}{\xi_i a}\right)^2\right]\right\}[\xi_i J'_m(\xi_i b) + k J_m(\xi_i b)]^2, \tag{4}$$

where ξ_i is a positive root of

$$[\xi_i Y'_m(\xi_i a) + h Y_m(\xi_i a)][\xi_i J'_m(\xi_i b) + k J_m(\xi_i b)] = [\xi_i Y'_m(\xi_i b) + k Y_m(\xi_i b)][\xi_i J'_m(\xi_i a) + h J_m(\xi_i a)], \tag{5}$$

$$\left\{ \begin{aligned} & H\left[\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} - \frac{m^2}{r^2} f\right] = \\ & = \frac{2}{\pi}\frac{\xi_i J'_m(\xi_i a) + h J_m(\xi_i a)}{\xi_i J'_m(\xi_i b) + k J_m(\xi_i b)}[f'(b) + k f(b)] - \frac{2}{\pi}[f'(a) + h f(a)] - \xi_i^2 \bar{f}(\xi_i), \end{aligned} \right. \tag{6}$$

where:

$J_m(\xi_i r)$, $Y_m(\xi_i r)$ = BESSEL functions of the first and second kind respectively and of order m;

h, k = constant coefficients whose value can be positive, negative or zero;

a, b = inner and outer radii of the shell respectively;

$f(r)$ = arbitrary function in the spatial variable r.

2. - Mathematical formulation of the problem and the initial and boundary conditions.

We choose cylindrical coordintes (r, θ, z) and assume for the forced torsional vibration the displacement components as

$$u_r = u_z = 0, \qquad u_\theta = u_\theta(r, z, t). \tag{7}$$

Assuming the rigidity and the density of the material to vary as the square of the distance from the axis of the cylinder, we have

$$\mu = \mu_0 r^2, \qquad \varrho = \varrho_0 r^2 \qquad (\mu_0, \ \varrho_0 \text{ constants}). \tag{8}$$

The only equation of motion that is not identically satisfied is

$$\frac{\partial^2 v}{\partial r^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{4}{r^2}\,v+\frac{\partial^2 v}{\partial z^2}=\frac{1}{c_2^2}\frac{\partial^2 v}{\partial t^2}\,, \tag{9}$$

where

$$v(r,\ z,\ t)=r\,u_\theta(r,\ z,\ t)\,,\qquad c_2^2=\frac{\mu_0}{\varrho_0}\,. \tag{10}$$

Initial conditions are

$$v=\frac{\partial v}{\partial t}=0,\qquad t=0\,,\qquad a\leqslant r\leqslant b\,. \tag{11}$$

The regularity conditions are

$$\operatorname*{Lt}_{z\to\pm\infty} v(r,\ z,\ t)=0\,, \tag{12}$$

and the boundary conditions are

$$[\widehat{r\theta}]_{r=a}=\mu_0\left(r\frac{\partial v}{\partial r}-2v\right)\bigg|_{r=a}=A(z,\ t)\qquad (r=a,\ t\geqslant 0), \tag{13}$$

$$[\widehat{r\theta}]_{r=b}=\mu_0\left(r\frac{\partial v}{\partial r}-2v\right)\bigg|_{r=b}=B(z,\ t)\qquad (r=b,\ t\geqslant 0), \tag{14}$$

where c_2 = (shear wave velocity of the material).

3. - Solution of the problem.

We assume in (6)

$$h=-\frac{2\mu_0}{a\mu_0}\,,\qquad k=-\frac{2\mu_0}{b\mu_0}\,, \tag{15}$$

then for $m=2$ we have

$$H\left[\frac{\mathrm{d}^2 f}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}f}{\mathrm{d}r}-\frac{2^2}{r^2}f\right]=\frac{2}{\pi a\mu_0}\left\{\frac{[\xi_i J_2'(\xi_i a)+h\,J_2(\xi_i a)]a}{[\xi_i J_2'(\xi_i b)+k\,J_2(\xi_i b)]b}\,\cdot\right.$$
$$\left.\cdot\,\mu_0\,[b\,f'(b)-2\,f(b)]-\mu_0\,[a\,f'(a)-2\,f(a)]\right\}-\xi_i^2\,\bar{f}(\xi_i)\,. \tag{16}$$

We choose the finite HANKEL transform in the form

$$\bar{v} = \bar{v}(\xi_i\,,\ z,\ t) = \int_a^b r\, v(r,\ z,\ t)\, C_2(r,\ \xi_i)\, \mathrm{d}r\,. \tag{17}$$

Applying transformation (17) to equation (9) and using equations (13), (14) and (16), one gets

$$\begin{cases} \dfrac{1}{c_2^2}\, \dfrac{\partial^2 v}{\partial t^2} = \dfrac{\partial^2 v}{\partial z^2} - \xi_i^2\, \bar{v}\, + \\[2ex] + \dfrac{2}{\pi a \mu_0} \left\{ \dfrac{[\xi_i\, J_2'(\xi_i\, a) + h\, J_2(\xi_i\, a)]a}{[\xi_i\, J_2'(\xi_i\, b) + k\, J_2(\xi_i\, b)]b} \cdot \widehat{r\theta}(b,\ z,\ t) - \widehat{r\theta}(a,\ z,\ t) \right\}. \end{cases} \tag{18}$$

Applying the surface loads from (13) and (14), gives

$$\begin{cases} \dfrac{1}{c_2^2}\, \dfrac{\partial^2 \bar{v}}{\partial t^2} + \xi_i^2\, \bar{v} = \\[2ex] = \dfrac{\partial^2 v}{\partial z^2} + \dfrac{2}{\pi \varrho_0\, c_2^2\, a} \left\{ \dfrac{[\xi_i\, J_2'(\xi_i\, a) + h\, J_2(\xi_i\, a)]a}{[\xi_i\, J_2'(\xi_i\, b) + k\, J_2(\xi_i\, b)]b} \cdot B(z,\ t) - A(z,\ t) \right\}. \end{cases} \tag{19}$$

For the axial variable z we choose the following FOURIER transform satisfying the regularity conditions (12),

$$\bar{\bar{v}} = \bar{\bar{v}}(\xi_i\,,\ \zeta,\ t) = \int_{-\infty}^{\infty} \bar{v}(\xi_i\,,\ z,\ t)\, e^{i\zeta z}\, \mathrm{d}z\,. \tag{20}$$

In view of (20), (19) becomes

$$\begin{cases} \dfrac{d^2 \bar{\bar{v}}}{dt^2} + c_2^2\, (\xi_i^2 + \zeta^2)\, \bar{\bar{v}} = \\[2ex] = \dfrac{2}{\pi a \varrho_0} \left\{ \dfrac{[\xi_i\, J_2'(\xi_i\, a) + h\, J_2(\xi_i\, a)]a}{[\xi_i\, J_2'(\xi_i\, b) + k\, J_2(\xi_i\, b)]b}\, \bar{B}(\zeta,\ t) - \bar{A}(\zeta,\ t) \right\}, \end{cases} \tag{21}$$

where

$$\begin{matrix} \bar{A}(\zeta,\ t) \\ \bar{B}(\zeta,\ t) \end{matrix} = \int_{-\infty}^{\infty} \begin{matrix} A(z,\ t) \\ B(z,\ t) \end{matrix}\, e^{i\zeta z}\, \mathrm{d}z\,. \tag{22}$$

Using LAPLACE transform and convolution integrals and the initial conditions (11) the solution to equation (21) can be written as

$$
(23)\quad \left\{\begin{aligned}
\bar{\bar{v}} &= \frac{2}{\pi\, a\, \varrho_0\, c_2} \int_0^t \left\{ \frac{[\xi_i\, J_2'(\xi_i\, a) + h\, J_2(\xi_i\, a)]a}{[\xi_i\, J_2'(\xi_i\, b) + k\, J_2(\xi_i\, b)]b}\, \bar{B}(\zeta,\ \tau) - \bar{A}(\zeta,\ \tau) \right\} \cdot \\
&\quad \cdot \frac{\sin c_2(t-\tau)\,(\xi_i^2 + \zeta^2)^{1/2}}{(\xi_i^2 + \zeta^2)^{1/2}}\, \mathrm{d}\zeta\,.
\end{aligned}\right.
$$

Using inverse FOURIER transform [3] we have

$$
(24)\quad \left\{\begin{aligned}
\bar{v} &= \frac{1}{\pi^2\, a\, \varrho_0\, c^2} \int_{-\infty}^{\infty} \int_0^t \left\{ \frac{[\xi_i\, J_2'(\xi_i\, a) + h\, J_2(\xi_i\, a)]a}{[\xi_i\, J_2'(\xi_i\, b) + k\, J_2(\xi_i\, b)]b}\, \bar{B}(\zeta,\ \tau) - \bar{A}(\zeta,\ \tau) \right\} \cdot \\
&\quad \cdot\ e^{-i\zeta z}\, \frac{\sin c_2(t-r)\,(\xi_i^2 + \varrho^2)^{1/2}}{(\xi_i^2 + \varrho^2)^{1/2}}\, \mathrm{d}\tau\, \mathrm{d}\varrho\,.
\end{aligned}\right.
$$

Applying inverse HANKEL transform from (3), we get

$$
(25)\qquad v(r,\ z,\ t) = \frac{\pi^2}{2} \sum \xi_i^2\, [\xi_i\, J_2'(\xi_i\, b) + k\, J_2(\xi_i\, b)]^2\ \bar{v}(\xi_i\,,\ z,\ t)\, \frac{C_2(r,\ \xi_i)}{F_2(\xi_i)}\,.
$$

Placing equation (24) in (25), we get

$$
(26)\quad \left\{\begin{aligned}
v(r,\ z,\ t) &= \frac{1}{2a\, \varrho_0\, c_2} \sum \xi_i^2\, [\xi_i\, J_2'(\xi_i\, b) + k\, J_2(\xi_i\, b)]^2\, \frac{C_2(r,\ \xi_i)}{F_2(\xi_i)} \cdot \\
&\quad \cdot \int_{-\infty}^{\infty} \int_0^t \left\{ \frac{[\xi_i\, J_2'(\xi_i\, a) + h\, J_2(\xi_i\, a)]a}{[\xi_i\, J_2'(\xi_i\, b) + k\, J_2(\xi_i\, b)]b}\, \bar{B}(\zeta,\ \tau) - \bar{A}(\zeta,\ \tau) \right\} \cdot \\
&\quad \cdot \frac{\sin c_2(t-\tau)\,(\xi_i^2 + \zeta^2)^{1/2}}{(\xi_i^2 + \zeta^2)^{1/2}} \cdot e^{-i\zeta z}\, \mathrm{d}\tau\, \mathrm{d}\zeta\,.
\end{aligned}\right.
$$

The stresses are then given by the formulae

$$
(27)\quad \begin{cases}
\widehat{r\theta} = \mu_0 \left(r \dfrac{\partial v}{\partial r} - 2v \right) = \\[2ex]
= \dfrac{c_2}{2a} \sum\limits_i \xi_i^2 \dfrac{[\xi_i J_2'(\xi_i b) + k J_2(\xi_i b)]^2}{F_2(\xi_i)} \left\{ r \dfrac{\partial}{\partial r} C_2(r, \xi_i) - 2 C_2(r, \xi_i) \right\} \cdot \\[2ex]
\cdot \int\limits_{-\infty}^{\infty} \int\limits_0^t \left\{ \dfrac{[\xi_i J_2'(\xi_i a) + h J_2(\xi_i a)]a}{[\xi_i J_2'(\xi_i b) + k J_2(\xi_i b)]b} \overline{B}(\zeta, \tau) - \overline{A}(\zeta, \tau) \right\} \cdot \\[2ex]
\cdot \dfrac{\sin c_2(t-\tau)(\xi_i^2 + \zeta^2)^{1/2}}{(\xi_i^2 + \zeta^2)^{1/2}} e^{-i\zeta z} \, d\tau \, d\zeta ,
\end{cases}
$$

$$
(28)\qquad \widehat{\theta z} = \mu_0 r^2 \frac{\partial u_0}{\partial z} = \mu_0 r \frac{\partial v}{\partial z} .
$$

4. - Evaluation of integrals.

The stresses $\widehat{r\theta}$ and $\widehat{\theta z}$ can be evaluated if the integrals in v in (26) can be evaluated in finite forms. We observe that the integrals there are of the type

$$
(29)\qquad I = \int\limits_0^t \int\limits_{-\infty}^{\infty} \overline{F}(\zeta, \tau) \frac{\sin c_2(t-\tau)(\xi_i^2 + \zeta^2)^{1/2}}{(\xi_i^2 + \zeta^2)^{1/2}} e^{-i\zeta z} \, d\tau \, d\zeta ,
$$

where

$$
(30)\qquad \overline{F}(\zeta, \tau) = \int\limits_{-\infty}^{\infty} F(z, \tau) e^{i\zeta z} \, dz ,
$$

and $F(z, \tau)$ is given by the prescribed boundary conditions on the surfaces $r = a$ and $r = b$.

Thus we see that the solution of the problem in the non-homogeneous case too depends on the evaluation of the same integral as got by CINELLI [2] and which he integrated in finite form in some particular cases.

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References.

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Abstract.

In this Note the torsional vibrations and stresses in an elastic cylinder characterised by a particular type of inhomogeneity have been discussed. In the solution of the problem finite Hankel transforms as well as Fourier and Laplace transforms have been used.

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