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Elements of Linear Polygenic Transformations and Pseudo-Angles of a Complex Vector Space. (**)

1. - Pseudo-conformal geometry.

It is well-known that the study of analytic functions of one complex variable is identical with that of conformal maps T. However, the extension to the analogous situation for two or more complex variables, is not related to the study of conformal correspondences T except in certain specialized cases. In 1907, POINCARÉ [1], termed a map T of analytic functions with non-vanishing Jacobian, a *regular transformation* T. However, from a geometrical point of view, KASNER found it more convenient to term such a correspondence a *pseudo-conformal transformation* T. Also, the study of the group of such correspondences T, was called *pseudo-conformal geometry* by the latter author. Presently, this is standard terminology.

The geometry of the pseudo-conformal group G of a pseudo-conformal space $\sum_{2n}$ of *finite* dimension $2n \geqslant 2$, was characterized by means of the pseudo-angle by KASNER [2] and also by KASNER and DECICCO [3, 4].

In the present article, pseudo-conformal geometry is extended to a complex vector space V, either finite or infinite dimensional, without the use of an inner product or metric.

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2. - Polygenic and pseudo-conformal linear transformations.

Let V denote a complex vector space, that is, V is a vector space for which the field of scalars is the complex number system [5]. Such a vector space is called a *contravariant vector space* V, and the vectors of this complex vector space V, are said to be *contravariant vectors*.

A single valued map T on the contravariant vector space V into a non-empty subset of V, is called a *polygenic linear transformation* T if and only if it possesses the following two properties, namely

$$(A) \qquad T(\lambda_1 + \lambda_2) = T(\lambda_1) + T(\lambda_2) \qquad \text{for all } \lambda_1, \ \lambda_2 \in V,$$

$$(B) \qquad T(z\,\lambda) = z\,T(\lambda) \qquad \text{for all } \lambda \in V, \ z \in R^{\#},$$

for every real scalar z of the real number system $R^{\#}$.

The concept of a polygenic function was introduced by KASNER in 1927 [6, 7, 8].

The extension to differentials of first order of a polygenic correspondence, is depicted as a polygenic linear transformation T.

A polygenic linear transformation T is termed either *direct pseudo-conformal* or *reverse pseudo-conformal* if and only if either $T(z\,\lambda) = z\,T(\lambda)$, or $T(z\,\lambda) = \bar{z}\,T(\lambda)$, for every finite complex number $z = x + iy$.

Theorem 2.1. *A polygenic linear transformation T is either direct pseudo-conformal or reverse pseudo-conformal if and only if either*

$$(2.1) \qquad T(i\,\lambda) = i\,T(\lambda) \qquad \textit{for all } \lambda \in V,$$

or

$$(2.2) \qquad T(i\,\lambda) = -\,i\,T(\lambda) \qquad \textit{for all } \lambda \in V.$$

This result follows directly from properties (A) and (B) and the definitions of direct and reverse pseudo-conformal.

Let R denote the totality of all polygenic linear transformations T, Γ^* be the set of all direct and reverse linear pseudo-conformal transformations T, and Γ be the set of all direct linear pseudo-conformal trasnformations T.

Theorem 2.2. *Relative to composition, each one of the three sets R, Γ^*, Γ, is a semi-group with identity I, such that Γ is a proper subset of Γ^*, and Γ^* is a* *proper subset of R. That is,*

$$\varphi \neq \Gamma \subset \Gamma^* \subset R\,. \tag{2.3}$$

Moreover, each of the two sets R and Γ is an associative ring with identity I relative to vector addition and composition.

Here, complex scalars are permissible in both Γ and Γ^*, but only real scalars are allowed in R. Thus, Γ is a complex vector space, wheras R is a real vector space.

Theorem 2.3. *For every polygenic linear transformation T, there is a unique direct linear pseudo-conformal transformation T_1 and a unique reverse linear pseudo-conformal transformation T_2, such that*

$$2\,T(\lambda) = T_1(\lambda) + T_2(\lambda). \tag{2.4}$$

Moreover,

$$\begin{cases} T_1(\lambda) &= T(\lambda) - i\,T(i\,\lambda) \\ T_2(\lambda) &= T(\lambda) + i\,T(i\,\lambda)\,. \end{cases} \tag{2.5}$$

For, it is clear that if there were such a decomposition, then

$$\begin{cases} 2\,T(\lambda) &= T_1(\lambda) + T_2(\lambda) \\ 2\,T(i\,\lambda) &= i\,T_1(\lambda) - i\,T_2(\lambda)\,. \end{cases} \tag{2.6}$$

Solving these for $T_1(\lambda)$ and $T_2(\lambda)$, (2.5) are obtained.

3. - Some applications.

The concept of a polygenic linear transformation T, may be applied to various parts of real and complex mathematical analysis.

The following is a list of five illustrations.

(I) Let C^n denote n-dimensional complex number space, and let f be a polygenic function on some open set of C^n, that is, the real and imaginary parts of f are continuously differentiable on the open set. Then the differential of f at some point of the region is

$$\text{(3.1)} \qquad \mathrm{d}f = \sum_{h=1}^{n} \frac{\partial f}{\partial z_h} \, \mathrm{d}z_h + \sum_{h=1}^{n} \frac{\partial f}{\partial \bar{z}_h} \, \mathrm{d}\bar{z}_h \, .$$

(See [**9, 10, 11, 12**].) This differential is a polygenic linear transformation.

(II) The HILBERT space l^2 is that of all sequences (a_n) of complex numbers such that

$$\sum_{n=0}^{\infty} | a_n |^2 < +\infty \, .$$

Then the map

$$\text{(3.2)} \qquad T(a_n) = (\bar{a}_n) \, ,$$

which transforms each sequence (a_n) of l^2 into the sequence whose terms are the complex conjugates of the corresponding ones of the original, is polygenic linear on l^2.

(III) Let $\mathscr{E}$ be the space of all infinitely-differentiable complex-valued functions f of a single real variable t. The map T on $\mathscr{E}$, defined by

$$\text{(3.3)} \qquad Tf = \frac{\mathrm{d}}{\mathrm{d}t} \bar{f}$$

for every f in $\mathscr{E}$, is a polygenic linear map T on $\mathscr{E}$.

(IV) In the space C of continuous complex-valued functions defined on a compact interval $[a, b]$ in $R^{\#}$, the map T for which

$$\text{(3.4)} \qquad Tf = g \, ,$$

where

$$\text{(3.5)} \qquad g(x) = \int_a^x f(t) \, \mathrm{d}t \, , \qquad a \leqslant x \leqslant b \, ,$$

is a polygenic linear transformation T on C.

(V) Consider the space V of all bounded linear operators on a complex HILBERT space H. Consider the map T such that

$$T(A) = A^* , \tag{3.6}$$

where A^* is the adjoint operator of A in V. By the elementary theory of adjoint operators [**13**, **14**, **15**], T is found to be polygenic linear on V.

Note that examples (II), (III), (IV), (V) are examples of reverse linear pseudo-conformal transformations, wheras example (I) is neither a direct nor a reverse linear pseudo-conformal transformation.

4. - The complex covariant vector space V^* of covariant vectors.

A polygenic linear functional (λ, μ) on a complex vector space V of contravariant vectors λ, is a polygenic linear transformation (λ, μ) for which the domain is the complex contravariant vector space V of contravariant vectors λ, and the range is a subset, proper or improper, of the complex number system.

It is assumed that the set of all polygenic linear functionals (λ, μ), is in one-to-one correspondence with a set V^* of elements μ.

By appropriately defining vector addition and complex scalar multiplication of covariant vectors μ, it follows that this set is a complex covariant vector space V^* of covariant vectors μ. (See [**16**].)

In addition, the linear functional (λ, μ) is a polygenic linear functional both in the contravariant vector λ and in the covariant vector μ.

Theorem 4.1. *A polygenic linear functional (λ, μ), is a single-valued complex function whose domain is the Cartesian product of the complex contravariant vector space V of contravariant vectors λ and of the complex covariant vector space V^* of covariant vectors μ, and whose range is a non-empty subset, proper or improper, of the complex number system. It is bilinear in the sense that it possesses the following two properties:*

(A) *If λ_1, λ_2, $\lambda \in V$ and μ_1, μ_2, $\mu \in V^*$, it obeys the two distributive laws*

$$\begin{cases} (\lambda_1 + \lambda_2 , \mu) = (\lambda_1 , \mu) + (\lambda_2 , \mu) \\ (\lambda, \mu_1 + \mu_2) = (\lambda, \mu_1) + (\lambda, \mu_2) . \end{cases} \tag{4.1}$$

(B) *It is linear homogeneous relative to the real number system* $R^{\#}$. *Thus for every contravariant vector* λ, *for every covariant vector* μ, *and for every finite real number* z, *it satisfies the linear homogeneous condition*

$$(4.2) \qquad (z\lambda, \mu) = (\lambda, z\mu) = z(\lambda, \mu).$$

A polygenic linear functional is said to be either *direct* or *reverse pseudo-conformal relative to the contravariant vector space* V, if and only if

$$(4.3) \qquad \text{either } (z\lambda, \mu) = z(\lambda, \mu) \quad \text{or} \quad (z\lambda, \mu) = \bar{z}(\lambda, \mu),$$

for every contravariant vector λ of the contravariant vector space V, and for every finite complex number $z = x + iy$.

Dually, a polygenic linear functional (λ, μ), is considered to be either *direct* or *reverse pseudo-conformal relative to the covariant vector space* V^*, if and only if

$$(4.4) \qquad \text{either } (\lambda, z\mu) = z(\lambda, \mu) \quad \text{or} \quad (\lambda, z\mu) = \bar{z}(\lambda, \mu),$$

for every covariant vector μ of the covariant vector space V^*, and for every finite complex number $z = x + iy$.

A *direct pseudo-conformal linear functional* $[\lambda, \mu]$, is a polygenic linear functional that is direct pseudo-conformal relative to the contravariant vector space V, and reverse pseudo-conformal relative to the covariant vector space V^*.

Dually, a *reverse pseudo-conformal linear functional* $\overline{[\lambda, \mu]}$ is a polygenic linear functional that is reverse pseudo-conformal relative to the contravariant vector space V, and direct pseudo-conformal relative to the covariant vector space V^*.

A polygenic linear functional is a direct pseudo-conformal linear functional if and only if

$$(4.5) \qquad [z\lambda, \mu] = z[\lambda, \mu] \quad \text{and} \quad [\lambda, z\mu] = \bar{z}[\lambda, \mu],$$

for every finite complex number $z = x + iy$.

It is clear that a reverse pseudo-conformal linear functional $\overline{[\lambda, \mu]}$, is the complex conjugate of a direct pseudo-conformal linear functional.

Theorem 4.2. *For any polygenic linear functional* (λ, μ), *there exists one and only one set of two direct pseudo-conformal linear functionals* $[\lambda, \mu]_1$ *and* $[\lambda, \mu]_2$, *such that*

$$(4.6) \qquad (\lambda, \mu) + (i\lambda, i\mu) = [\lambda, \mu]_1 + \overline{[\lambda, \mu]_2}.$$

Moreover,

$$(4.7)\qquad \begin{cases} 2\,[\lambda,\ \mu]_1 = (\lambda,\ \mu) - i\,(i\,\lambda,\ \mu) = i\,(\lambda,\ i\,\mu) + (i\,\lambda,\ i\,\mu) \\ 2\,\overline{[\lambda,\ \mu]_2} = (\lambda,\ \mu) + i\,(i\,\lambda,\ \mu) - i\,(\lambda,\ i\,\mu) + (i\,\lambda,\ i\,\mu)\,. \end{cases}$$

For,

$$(4.8)\qquad \begin{cases} (\lambda,\ \mu) + (i\,\lambda,\ i\,\mu) = [\lambda,\ \mu]_1 + \overline{[\lambda,\ \mu]_2} \\ (i\,\lambda,\ \mu) - (\lambda,\ i\,\mu) = i\,[\lambda,\ \mu]_1 - i\,\overline{[\lambda,\ \mu]_2}\,. \end{cases}$$

Solving these for $[\lambda,\ \mu]_1$ and $[\lambda,\ \mu]_2$, the equations (4.7) are found. Also, it is obvious that $[\lambda,\ \mu]_1$ and $[\lambda,\ \mu]_2$ are two direct pseudo-conformal linear functionals.

A polygenic linear functional $(\lambda,\ \mu)$ is said to possess the *conjugate-symmetric property* if and only if

$$(4.9)\qquad (i\,\lambda,\ \mu) = -(\lambda,\ i\,\mu)\,.$$

For a conjugate-symmetric polygenic linear functional $(\lambda,\ \mu)$, it is seen that

$$(4.10)\qquad (i\,\lambda,\ i\,\mu) = (\lambda,\ \mu)\,.$$

Theorem 4.3. *For any polygenic linear functional* $(\lambda,\ \mu)$ *there corresponds a conjugate-symmetric polygenic linear functional* $\lambda\cdot\mu$, *namely,*

$$(4.11)\qquad \lambda\cdot\mu = \frac{1}{2}\{(\lambda,\ \mu) + (i\,\lambda,\ i\,\mu)\}\,.$$

For a conjugate-symmetric polygenic linear functional $\lambda\cdot\mu$ *the corresponding set of two direct pseudo-conformal functionals* $[\lambda,\ \mu]_1$ *and* $[\lambda,\ \mu]_2$ *are*

$$(4.12)\qquad \begin{cases} [\lambda,\ \mu]_1 = \lambda\cdot\mu - i\,(i\,\lambda\cdot\mu) = \lambda\cdot\mu + i\,(\lambda\cdot i\,\mu) \\ \overline{[\lambda,\ \mu]_2} = \lambda\cdot\mu + i\,(i\,\lambda\cdot\mu) = \lambda\cdot\mu - i\,(\lambda\cdot i\,\mu)\,. \end{cases}$$

For, it is evident that (4.11) obeys the condition (4.9). Also by (4.9) and (4.10), the relations (4.7) become the relations (4.12).

Theorem 4.4. *If a conjugate-symmetric linear functional $\lambda\cdot\mu$ is real, then the corresponding set of two direct pseudo-conformal linear functionals consists of one and only one direct pseudo-conformal linear functional $[\lambda,\ \mu]$. Also*

$$(4.13)\quad \begin{cases} \lambda\cdot\mu = \frac{1}{2}\{[\lambda,\ \mu] + \overline{[\lambda,\ \mu]}\}, \qquad \lambda\cdot i\,\mu = -(i\,\lambda\cdot\mu) = \frac{1}{2i}\{[\lambda,\ \mu] - \overline{[\lambda,\ \mu]}\} \\ [\lambda,\ \mu] = \lambda\cdot\mu + i\,(\lambda\cdot i\,\mu) = \lambda\cdot\mu - i\,(i\,\lambda\cdot\mu)\,. \end{cases}$$

This is a consequence of Theorem 4.3.

It is remarked that a direct pseudo-conformal linear functional $[\lambda,\ \mu]$ is an abstraction of the inner product of a complex inner product space [5].

5. - The pseudo-angle θ.

A complex contravariant vector space V of vectors λ and its dual complex covariant vector space V^* of vectors μ, are said to be *pseudo-conformal* if and only if there can be associated a definite direct pseudo-conformal linear functional $[\lambda,\ \mu]$ with the two properties

$$(5.1)\qquad [z\,\lambda,\ \mu] = z\,[\lambda,\ \mu]\,, \qquad [\lambda,\ z\,\mu] = \bar{z}\,[\lambda,\ \mu]\,,$$

for every finite complex number $z = x + i\,y$. Then if

$$\lambda\cdot\mu = \tfrac{1}{2}\{[\lambda,\ \mu] + \overline{[\lambda,\ \mu]}\}\,,$$

it follows that

$$(5.2)\quad \begin{cases} \lambda\cdot\mu = \frac{1}{2}\{[\lambda,\ \mu] + \overline{[\lambda,\ \mu]}\}\,, \qquad \lambda\cdot i\,\mu = -(i\,\lambda\cdot\mu) = \frac{1}{2i}\{[\lambda,\ \mu] - \overline{[\lambda,\ \mu]}\} \\ [\lambda,\ \mu] = \lambda\cdot\mu + i\,(\lambda\cdot i\,\mu) = \lambda\cdot\mu - i\,(i\,\lambda\cdot\mu)\,. \end{cases}$$

This is the *real conjugate-symmetric linear functional* $\lambda\cdot\mu$ of a pseudo-conformal complex contravariant vector space V and of the dual pseudo-conformal complex covariant vector space V^*.

If in the complex contravariant vector space V, pseudo-conformal or not, $\lambda \neq 0$, is a non-zero contravariant vector, then the set of all contravariant vectors $\nu = \varrho\lambda$, where $\varrho = \varrho_1 + i\varrho_2$, is a finite complex number, is said to describe an *isocline plane* π_2.

If $\lambda \neq 0$, and $\nu = \varrho\, \lambda \neq 0$, where $\varrho \neq 0$, is a finite complex number, are two contravariant vectors in the same isocline plane π_2, then the angle θ, with $0 \leqslant \theta < 2\pi$, for which $\varrho = |\varrho| \exp(i\theta) \neq 0$, is called the *angle* θ or the *pseudo-angle* θ from the vector $\lambda \neq 0$ to the vector $\nu = \varrho\, \lambda \neq 0$.

This angle θ obeys the relation

$$\bar{\varrho}/\varrho = \exp(-2i\theta), \qquad \text{where } \varrho \neq 0\,. \tag{5.3}$$

In the pseudo-conformal complex contravariant vector space V and in the dual pseudo-conformal complex contravariant vector space V^*, a contravariant vector λ and a covariant vector μ, are said to be *transversal* if and only if

$$\lambda\cdot\mu = \frac{1}{2}\{[\lambda,\, \mu] + \overline{[\lambda,\, \mu]}\} = 0\,. \tag{5.4}$$

In general, this relation of transversality is *not* symmetric. This is an abstraction of the transversality in the calculus of variations [**17**, **18**].

If $\mu \neq 0$ is a fixed non-zero covariant vector, then the set of all contravariant vectors λ transversal to μ is termed the *transversal complement* of μ.

Similarly, if $\lambda \neq 0$ is a given non-zero contravariant vector, then the set of all covariant vectors μ, transversal to λ, is said to be the *transversal complement* of λ.

The transversal complement of a fixed covariant vector $\mu \neq 0$, or of a given contravariant vector $\lambda \neq 0$, is either a contravariant or covariant complex vector space of deficiency one, which is a proper subspace of either the contravariant or covariant vector space V or V^*.

If $\lambda \neq 0$ is a given contravariant vector, then a non-zero contravariant vector in the same isocline plane π_2 with λ, is of the form $\nu = |\varrho| \exp(-i\theta)\, \lambda \neq 0$, where $|\varrho| > 0$ and θ, with $0 \leqslant \theta < 2\pi$, are two real numbers. Of course, θ is the angle from the new vector $\nu \neq 0$, to the given vector λ.

This new contravariant vector $\nu = |\varrho| \exp(-i\theta)\, \lambda \neq 0$, is in the transversal complement of a given non-zero covariant vector $\mu \neq 0$, if and only if

$$\exp(-i\,\theta)\,[\lambda,\, \mu] + \exp(i\,\theta)\,\overline{[\lambda,\, \mu]} = 0\,. \tag{5.5}$$

This is so if and only if $[\lambda,\, \mu] = i\,\varrho \exp(i\,\theta)$, where ϱ is a finite real number.

The angle θ, with $0 \leqslant \theta < 2\pi$, if it exists, of equation (5.5), is termed the *pseudo-angle* θ from the contravariant vector $\lambda \neq 0$, to the covariant vector $\mu \neq 0$.

Consequently, the following result is obtained:

Theorem 5.1. *In a pseudo-conformal complex contravariant vector space V and in its dual pseudo-conformal complex covariant vector space V^*, an angle θ, with $0 \leqslant \theta < 2\pi$ is a pseudo-angle θ from a non-zero contravariant vector λ to a non-zero covariant vector $\mu \neq 0$, if and only if*

$$[\lambda, \mu] = i \varrho \exp(i \theta) , \tag{5.6}$$

where ϱ is a finite real number. This is equivalent to saying that

$$\lambda \cdot \mu = - \varrho \sin\theta, \qquad \lambda \cdot (i \mu) = - (i \lambda) \cdot \mu = \varrho \cos\theta. \tag{5.7}$$

This pseudo-angle θ, is indeterminate if and only if $[\lambda, \mu] = 0$. This means geometrically that the isocline plane π_2 determined by the contravariant vector $\lambda \neq 0$, is in the transversal complement of the covariant vector $\mu \neq 0$.

If $[\lambda, \mu] \neq 0$, this pseudo-angle θ, with $0 \leqslant \theta < 2\pi$, is determined uniquely if and only if $\varrho = | [\lambda, \mu] | > 0$.

Consider a contravariant vector $\lambda \neq 0$, and a covariant vector $\mu \neq 0$, for which $[\lambda, \mu] \neq 0$. The pseudo-angle θ between them is $\pi/2$ radians, if and only if either λ and $i \mu$, or $i \lambda$ and μ, are transversal. Also, the pseudo-angle θ between $\lambda \neq 0$, and $\mu \neq 0$, is equal to the pseudo-angle θ between $i \lambda \neq 0$, and $i \mu \neq 0$.

6. - Transformation theory of polygenic linear transformations.

Let $(\lambda_1 , \mu_1)_1$ and $(\lambda_2 , \mu_2)_2$ be two polygenic linear functionals, for each of which the domain is the Cartesian product of the complex contravariant vector space V and the dual complex covariant vector space V^*, and the range is a non-empty proper or improper subset of the complex number system.

There may exist two contravariant vectors λ_1 and λ_2 and two covariant vectors μ_1 and μ_2 , such that

$$(\lambda_1 , \mu_1)_1 = (\lambda_2 , \mu_2)_2 . \tag{6.1}$$

If λ_2 corresponds to λ_1 by a linear transformation $\lambda_2 = T(\lambda_1)$ on the contravariant vectors λ of the complex contravariant vector space V, then there is nduced one and only one linear transformation $\mu_1 = T'(\mu_2)$ on the covariant vectors μ of the dual complex covariant vector space V^*, such that

$$(\lambda_1 , T(\mu_2))_1 = (T(\lambda_1) , \mu_2)_2 , \tag{6.2}$$

and conversely.

Two such linear transformations $\lambda_2 = T(\lambda_1)$ and $\mu_1 = T'(\mu_2)$, are said to be *transposes* of one another.

Theorem 6.1. *The operation of transposition between the ring of linear transformations $\lambda_2 = T(\lambda_1)$ on the contravariant vectors λ of a complex contravariant vector space V, and the ring of linear transformations $\mu_1 = T'(\mu_2)$ on the covariant vectors μ of the dual complex covariant vector space V^* is an anti-isomorphism. That is, the transpose $\mu_1 = T'(\mu_2)$ of $\lambda_2 = T(\lambda_1)$ possesses the following two properties* (A) *and* (B):

$$(T_1 + T_2)' = T_1' + T_2', \tag{A}$$

$$(T_2\, T_1)' = T_1'\, T_2'. \tag{B}$$

The derivation of property (A) is as follows. Since

$$\begin{cases} (\lambda_1\,,\ T_1'(\mu_2))_1 = (T(\lambda_1),\ \mu_2)_2 \\ (\lambda_1\,,\ T_2'(\mu_2))_1 = (T_2(\lambda_1)\,,\ \mu_2)_2\,, \end{cases} \tag{6.3}$$

then

$$(\lambda_1\,,\ (T_1' + T_2')(\mu_2))_1 = ((T_1 + T_2)(\lambda_1),\ \mu_2)_2\,. \tag{6.4}$$

Thus it follows that $(T_1 + T_2)' = T_1' + T_2'$.

For the proof of property (B), suppose $\lambda_2 = T_1(\lambda_1)$. Then $\mu_1 = T_1'(\mu_2)$, with

$$(\lambda_1\,,\ T_1'(\mu_2))_1 = (T_1(\lambda_1),\ \mu_2)_2\,. \tag{6.5}$$

If $\lambda_3 = T_2(\lambda_2)$, then $\mu_2 = T_2'(\mu_3)$, where

$$(\lambda_2\,,\ T_2'(\mu_3))_1 = (T_2(\lambda_2),\ \mu_3)_2\,. \tag{6.6}$$

Since $\lambda_3 = T_2(\lambda_2) = T_2\, T_1(\lambda_1)$, it is seen that $\mu_1 = T_1'(\mu_2) = T_1'\, T_2'(\mu_3)$. That is, $(T_2\, T_1)' = T_1'\, T_2'$.

A linear transformation $\lambda_2 = T(\lambda_1)$, on the contravariant vectors λ of a complex contravariant vector space V, is *non-singular* if and only if it has an inverse $\lambda_1 = T^{-1}(\lambda_2)$, such that $T^{-1}\, T = TT^{-1} = I$, the identity.

Theorem 6.2. *A linear transformation $\lambda_2 = T(\lambda_1)$, on the vectors λ of a complex contravariant vector space V, is non-singular if and only if its transpose $\mu_1 = T'(\mu_2)$ on the vectors μ of its dual complex covariant vector space V^*, is non-singular.*

For, if $\lambda_1 = I(\lambda_1)$ is the identity on the contravariant vectors λ of V, then

$$(6.7) \qquad (\lambda_1, I'(\mu_2))_1 = (I(\lambda_1), \mu_2)_2 = (\lambda_1, \mu_2)_2 .$$

Therefore, the transpose I' of the identity I, is the identity I itself.

Since $\lambda_2 = T(\lambda_1)$ is non-singular, then $T^{-1}T = TT^{-1} = I$. From Theorem 6.1, it follows that

$$(6.8) \qquad I'I(T^{-1}T)' = T'(T')^{-1} = (T')^{-1} T' .$$

That is, the transpose $\mu_1 = T'(\mu_2)$ of the non-singular linear transformation $\lambda_2 = T(\lambda_1)$, is non-singular.

Similarly, if $\mu_1 = T'(\mu_2)$ is non-sigular, then $\lambda_2 = T(\lambda_1)$ is non-singular.

7. - The direct pseudo-conformal group G and the total pseudo-conformal group G^*.

The *direct pseudo-conformal group* G is composed of all non-singular direct linear pseudo-conformal transformations T, on the contravariant vectors λ of a complex contravariant vector space V.

Similarly, the *total pseudo-conformal group* G^* consists of all non-singular direct and reverse pseudo-conformal transformations T on the contravariant vectors λ of a complex contravariant vector space V.

Neither one of these two groups G and G^* is empty, and G is a proper subgroup of G^*. That is, $\phi \neq G \subset G^*$, but $G \neq G^*$.

Let us consider the direct pseudo-conformal linear functional $[\lambda, \mu]$ of a pseudo-conformal complex contravariant vector space V of vectors λ, and of its dual pseudo-conformal complex covariant vector space V^* of vectors μ.

By a linear transformation $\lambda_2 = T(\lambda_1)$ and its transpose $\mu_1 = T'(\mu_2)$, the direct pseudo-conformal linear functional $[\lambda_1, \mu_1]$ corresponds to a polygenic linear functional $[\lambda_2, \mu_2]'$, *not* necessarily either direct or reverse pseudo-conformal, such that

$$(7.1) \qquad [\lambda_1, T'(\mu_2)] = [T(\lambda_1), \mu_2]'.$$

If T is non-singular, so is its transpose T'. Hence (7.1) can be written as

$$[\lambda_1, \mu_1] = [T(\lambda_1), (T')^{-1}(\mu_1)]'. \tag{7.2}$$

Theorem 7.1. *Under a non-singular linear transformation T and under its non-singular linear transpose T', the direct pseudo-conformal linear functional $[\lambda_1, \mu_1]$ is converted either into a direct pseudo-conformal linear functional $[\lambda_2, \mu_2]$, or into a reverse pseudo-conformal linear functional $\overline{[\lambda_2, \mu_2]}$, if and only if $\lambda_2 = T(\lambda_1)$ and $\mu_1 = T'(\mu_2)$, are both either direct pseudo-conformal or reverse pseudo-conformal.*

For, by equation (7.2), it is seen that

$$\begin{cases} [\lambda_1, \mu_1] = [T(\lambda_1), (T')^{-1}(\mu_1)]' \\ [T(i\,\lambda_1), \mu_2]' = i\,[T(\lambda_1), \mu_2]' \\ [\lambda_2, (T')^{-1}(i\mu_1)]' = -i\,[\lambda_2, (T')^{-1}(\mu_1)]'. \end{cases} \tag{7.3}$$

If $[\lambda_2, \mu_2] = [\lambda_2, \mu_2]' = [T(\lambda_1), T^{-1}(\mu_1)]'$ is a direct pseudo-conformal linear functional, then it follows that $T(i\,\lambda_1) = i\,T(\lambda_1)$ and $(T')^{-1}(i\,\mu_1) = i\,(T')^{-1}(\mu_1)$. That is, $\lambda_2 = T(\lambda_1)$ and $\mu_1 = T'(\mu_2)$, are both direct pseudo-conformal.

If $\overline{[\lambda_2, \mu_2]} = [\lambda_2, \mu_2]' = [T(\lambda_1), (T')^{-1}(\mu_1)]'$ is a reverse pseudo-conformal linear functional, then it is deduced that $T(i\lambda_1) = -i\,T(\lambda_1)$ and $(T')^{-1}(i\,\mu_1) = -i\,(T')^{-1}(\mu_1)$, are both reverse pseudo-conformal.

Theorem 7.2. *A non-degenerate linear transformation T on the contravariant vectors λ of a complex contravariant vector space V, is direct or reverse pseudo-conformal if and only if it carries every two contravariant vectors of an isocline plane π_2 into two contravariant vectors of an isocline plane π_2.*

For, the requirement of this proposition implies that the two contravariant vectors $\lambda \neq 0$, and $i\,\lambda \neq 0$, of an isocline plane π_2 must be carried into two contravariant vectors $T(\lambda)$ and $T(i\,\lambda)$ of an isocline plane π_2. Hence

$$T(i\,\lambda) = w\,T(\lambda), \tag{7.4}$$

for some finite non-zero complex number $w = u + iv \neq 0$.

Upon replacing $i\,\lambda$ by $-i\,\lambda$, it is seen that $T(i\,\lambda) = -w'T(\lambda)$, where $w' = u' + iv' \neq 0$, is some finite non-zero complex number. Hence $w = -w'$, so that $u = u' = 0$, and $v = v'$. Thus (7.4) becomes

$$T(i\lambda) = iv\, T(\lambda)\,, \tag{7.5}$$

where $v \neq 0$ is some finite real number.

If $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$ are two linearly independent vectors, then $T(i\lambda_1) = iv_1 T(\lambda_1)$ and $T(i\lambda_2) = iv_2 T(\lambda_2)$. Hence

$$T(i\lambda_1 + i\lambda_2) = iv_3\{T(\lambda_1) + T(\lambda_2)\} = iv_1 T(\lambda_1) + iv_2 T(\lambda_2)\,.$$

Since $T(\lambda_1) \neq 0$ and $T(\lambda_2) \neq 0$ are linearly independent, it follows that $v_3 = v_1 = v_2$.

Consequently the finite real number $v \neq 0$, of (7.5), is independent of the contravariant vector λ.

Therefore

$$T(i\lambda) = iv\, T(\lambda)\,, \qquad T(\lambda) = -iv\, T(i\lambda) = v^2 T(\lambda)\,. \tag{7.6}$$

That is, $v^2 = 1$, so that $v = +1$ or $v = -1$.

If $v = +1$, then $T(i\lambda) = i\,T(\lambda)$, so that T is direct pseudo-conformal. If $v = -1$, then $T(i\lambda) = -i\,T(\lambda)$, so that T is reverse pseudo-conformal.

The sufficiency of this proposition is evident.

Theorem 7.3. *A non-degenerate linear transformation T on the contravariant vectors λ of a complex contravariant vector space V, is direct or reverse pseudo-conformal if and only if its non-degenerate linear transpose T' on the covariant vectors μ of its dual complex covariant vector space V^*, is direct or reverse pseudo-conformal.*

For, if $\lambda \neq 0$ is a contravariant vector and $\mu \neq 0$ is a covariant vector, then $i\lambda \neq 0$ is orthogonal to $\mu \neq 0$ if and only if $[\lambda, \mu] = \overline{[\lambda, \mu]}$. Hence for every finite complex number $z = x + iy$, $iz\lambda$ is orthogonal to $z\mu$ since $[z\lambda, z\mu] = \overline{[z\lambda, z\mu]}$. This means that the isocline plane π_2 determined by the contravariant vector $i\lambda \neq 0$, is orthogonal to the isocline plane π_2' determined by the covariant vector μ.

Since a non-degenerate linear transformation T on the contravariant vectors λ of V is direct or reverse pseudo-conformal if and only if it converts every isocline plane of contravariant vectors into an isocline plane of contravariant vectors, then its non-degenerate linear transpose T' on the covariant vectors μ of the dual space V^*, carries every isocline plane of covariant vectors into an isocline plane of covariant vectors. By the dual of Theorem 7.2, the non-degenerate linear transpose T' is either direct or reverse pseudo-conformal.

Clearly the preceeding argument may be dualized. Hence Theorem 7.3 is established.

Of course, a non-degenerate linear transformation T on the contravariant vectors λ of V, is either direct or reverse pseudo-conformal according as its linear transpose T' on the covariant vectors of its dual space V^* is either direct or reverse pseudo-conformal.

Theorem 7.4. *A non-degenerate linear transformation T on the contravariant vectors λ of a complex contravariant vector space V, or its non-degenerate linear transpose T' on the covariant vectors μ of the dual complex covariant vector space V^*, is either direct or reverse pseudo-conformal if and only if it preserves the pseudo-angle θ, with $0 \leqslant \theta < 2\pi$, between every non-zero contravariant vector $\lambda \neq 0$ and every non-zero covariant vector $\mu \neq 0$. It is direct or reverse according as the orientation of the pseudo-angle θ, is preserved or reversed.*

For, by the method in which the pseudo-angle θ was constructed, such a linear map carries every isocline plane of either contravariant or covariant vectors into an isocline plane of either contravariant or covariant vectors. Thus such a map is necessarily only direct or reverse pseudo-conformal.

By Theorem 7.3, T and T' are both either direct or reverse pseudo-conformal.

By Theorem 5.1 and Theorem 7.1, the magnitude of the pseudo-angle θ, with $0 \leqslant \theta < 2\pi$, is preserved under the total pseudo-conformal group G^*. Evidently its orientation is preserved or reversed according as the map is either direct or reverse pseudo-conformal.

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Summary.

The theory of polygenic functions and that of pseudo-conformal geometry, originally due to Kasner, are extended to a complex vector space V, and its dual V^*, either finite or infinite dimensional. Polygenic and pseudo-conformal linear transformations are introduced. The theory of pseudo-conformal functionals is developed. Isocline planes in the contravariant space V and in the covariant space V^* are defined, the pseudo-angle θ between vectors in the same isocline plane is defined, and the theory of pseudo-angles θ between contravariant and covariant vectors is studied. The transformation theory of polygenic linear functionals is developed. The pseudo-conformal group G^* and the direct pseudo-conformal group G are characterized by the pseudo-angle θ.

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